THE ZERO SET OF A POLYNOMIAL

RICHARD CARON AND TIM TRAYNOR

The following result is intuitively obvious, and is accessible to students of a first course in Measure Theory, but we have been unable to find even the statement in publications at that level. A version for analytic functions with values in an arbitrary Banach space, relying on the concept of approximate differentiation, can be found in Federer's book[Fed69].

Theorem. *A polynomial function on $\mathbb{R}^n$ to $\mathbb{R}$, is either identically 0, or non-zero almost everywhere.*

Proof. By induction on n. We denote n-dimensional Lebesgue measure by $\boldsymbol{\lambda}_n$.

If $n = 1$ and p is a polynomial of degree m other than the zero polynomial, p has at most m roots, so $\boldsymbol{\lambda}_1\{x : p(x) = 0\} = 0$.

Now, suppose the result is true for polynomials in $n-1$ variables and let

$$\begin{aligned} p(x) = p(x_1, x_2, \dots, x_n) &= \sum_{k_1,k_2,\dots,k_n} a_{k_1k_2\dots k_n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \\ &= \sum_k a_k x^k \text{ (in multiindex notation)} \end{aligned}$$

For a multiindex $k = (k_1, k_2, \dots, k_n)$, write $k = (i, j)$, where $i = k_1$, $j = (k_2, \dots, k_n)$. Identifying $\mathbb{R}^n$ with $\mathbb{R} \times \mathbb{R}^{n-1}$, we can write

$$p(x) = p(x_1, \bar{x}_2) = \sum_j q_j(x_1) x^j,$$

where $q_j(x_1) = \sum_i a_{ij} x_1^i$.

Since p is not the zero polynomial on $\mathbb{R}^n$, there is a $j = (k_2, \dots, k_n)$, for which the polynomial function q_j is not identically 0. For such a j, $\{x_1 : q_j(x_1) = 0\}$ is finite; hence $N := \{x_1 : p(x_1, \bar{x}_2) = 0, \text{ for all } \bar{x}_2\}$ is also finite, so of measure 0.

On the other hand, for each fixed $x_1 \notin N$, the polynomial function $p_{x_1} = p(x_1, \cdot)$ is non-zero almost everywhere, by the inductive hypothesis. Thus,

$$\begin{aligned} \boldsymbol{\lambda}_n(\{x : p(x) = 0\}) &= \int \boldsymbol{\lambda}_{n-1}\{x_2 : p(x_1, \bar{x}_2) = 0\}\, dx_1 \\ &= \int_N \boldsymbol{\lambda}_{n-1}\{x_2 : p(x_1, \bar{x}_2) = 0\}\, dx_1 + \int_{N^c} 0\, dx_1 \\ &= 0 \end{aligned}$$

□

Date: 23 May 2005.

Supported by. . .

References

[Fed69] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR MR0257325 (41 #1976)

E-mail address: rcaron@uwindsor.ca and tt@uwindsor.ca

Department of Mathematics and Statistics, University of Windsor, Windsor Ontarion, N9B 3P4