THE ZERO SET OF A POLYNOMIAL

RICHARD CARON AND TIM TRAYNOR

The following result is intuitively obvious, and is accessible to students of a first course in Measure Theory, but we have been unable to find even the statement in publications at that level. A version for analytic functions with values in an arbitrary Banach space, relying on the concept of approximate differentiation, can be found in Federer’s book [Fed69].

**Theorem.** A polynomial function on $\mathbb{R}^n$ to $\mathbb{R}$, is either identically 0, or non-zero almost everywhere.

**Proof.** By induction on $n$. We denote $n$-dimensional Lebesgue measure by $\lambda_n$.

If $n = 1$ and $p$ is a polynomial of degree $m$ other than the zero polynomial, $p$ has at most $m$ roots, so $\lambda_1 \{ x : p(x) = 0 \} = 0$.

Now, suppose the result is true for polynomials in $n-1$ variables and let $p(x) = p(x_1, x_2, \ldots, x_n) = \sum_{k_1, k_2, \ldots, k_n} a_{k_1, k_2, \ldots, k_n} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$

For a multiindex $k = (k_1, k_2, \ldots, k_n)$, write $k = (i, j)$, where $i = k_1, j = (k_2, \ldots, k_n)$.

Identifying $\mathbb{R}^n$ with $\mathbb{R} \times \mathbb{R}^{n-1}$, we can write

$$p(x) = p(x_1, \bar{x}_2) = \sum_j q_j(x_1) x^j,$$

where $q_j(x_1) = \sum_i a_{ij} x_1^i$.

Since $p$ is not the zero polynomial on $\mathbb{R}^n$, there is a $j = (k_2, \ldots, k_n)$, for which the polynomial function $q_j$ is not identically 0. For such a $j$, $\{ x_1 : q_j(x_1) = 0 \}$ is finite; hence $N := \{ x_1 : p(x_1, \bar{x}_2) = 0, \text{ for all } \bar{x}_2 \}$ is also finite, so of measure 0.

On the other hand, for each fixed $x_1 \notin N$, the polynomial function $p_{x_1} = p(x_1, \cdot)$ is non-zero almost everywhere, by the inductive hypothesis. Thus,

$$\lambda_n \{ x : p(x) = 0 \} = \int N \lambda_{n-1} \{ x_2 : p(x_1, x_2) = 0 \} dx_1$$

$$= \int_N \lambda_{n-1} \{ x_2 : p(x_1, x_2) = 0 \} dx_1 + \int_{N^c} 0 dx_1$$

$$= 0$$

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References


E-mail address: rcaron@uwindsor.ca and tt@uwindsor.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WINDSOR, WINDSOR ONTARIO, N9B 3P4