A reconsideration of the Kar solution for minimum cost spanning tree problems

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Abstract

Minimum cost spanning tree (mcst) problems try to connect agents efficiently to a source when agents are located at different points in space and the cost of using an edge is fixed. The application of the Shapley value to the stand-alone cost game, known in this context as the Kar solution, has been criticized for sometimes proposing allocations that are outside of the core. I show that the situations where the Kar solution might be outside of the core are conceptually similar to glove-market games, which led to a criticism of the core by Shapley and Shubick (1969), as the notions of stability and fairness clash.

1 Introduction

Minimum cost spanning tree (mcst) problems model a situation where agents are located at different points and need to be connected to a source to obtain a good. Cost depends only on which edges are built and not on the number of users on each edge. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks. The common cost of construction must then be split among the participants.

There are (at least) two interpretations of the game. The private property approach is such that a group can only use the locations of its members to connect to the source, while the common property approach allows the use of any location to connect to the source. Bird (1976) introduced the Shapley value of the stand-alone cost game associated to the private property approach. It has been extensively studied in Kar (2002). This value (known as the Kar solution) has been largely criticized and dismissed for mcst problems as it sometimes proposes allocations that lie outside of the stand-alone core, meaning that some coalitions can secede from the group and connect to the source at a cost smaller than the shares they would pay under the Kar solution.

Bird (1976), Dutta and Kar (2004) and Bergantinos and Vidal-Puga (2007) use that objection to propose different cost sharing solutions (respectively the Bird, Dutta-Kar and folk solutions). In this paper, I take a closer look at that supposed weakness. I show that the cases where the Shapley value can prescribe allocations outside of the core are similar to those that lead to the criticism of the core by Shapley and Shubik (1969). In essence, the core predicts that if an agent has the same gain if he cooperates with disjoint groups S or T, he will be able to extract all of the surplus. In the private property approach, this is unfair to S and T, who are needed for the creation of a surplus. This prediction of the core also neglects the possibility that S and T will collude to keep part of the surplus.

2 Model and definitions

Let $N = \{1, ..., n\}$ be the set of agents and let 0 denote the source to which agents have to be connected. Let $N_0 = N \cup \{0\}$. A cost matrix c assigns a cost to all non-ordered pairs in N_0, with c_{ij} being the cost of edge (i,j). An elementary cost matrix is such that for all edges e, $c_e \in \{0,1\}$. A minimum cost spanning tree problem is (N,c), the set of players and the cost matrix.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_k, i_{k+1}), with $k \in [0, K-1]$ and such that $i_0 = i_K = l$ and $i_1, ..., i_{K-1}$ distinct and different than l. A path p_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k = 0, 1, ..., K-1$, containing no cycle and such that $i_0 = l$ and $i_K = m$ and $i_1, ..., i_{K-1}$ distinct and different than l and m. For any $S \supseteq \{l, m\}$, let $P_{lm}(S)$ be the set of all paths from l to m for which all edges are in S.

The minimum cost of connecting N to the source and the associated set of edges, called a minimum cost spanning tree, is obtained through Prim's algorithm which has $|N|$ steps. First, pick an edge $(0,i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. At each following step we connect an agent not already connected to the source or to an agent previously connected, until all agents are connected. Let $C(N,c)$ be the associated cost.

Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S,c)$ be the cost of the mcst of the problem (S, c^S). We say that C is the stand-alone cost function associated with c.

The irreducible cost matrix $\bar{c}$ was first defined in Bird (1976) and is obtained by reducing the cost of the edges as much as possible without changing the total cost of the project. It is defined by finding the path between i and j that has the cheapest most expensive edge and assigning that cost to edge (i,j). Formally, for each edge (i,j), we have

$$\bar{c}_{ij} = \min_{p_{ij} \in P_{ij}(N_0)} \max_{e \in p_{ij}} c_e,$$

For elementary matrices, $\bar{c}_{ij}$ has a special meaning, as it is the cost of the cheapest path from i to j.

A cost allocation $y \in \mathbb{R}^n$ assigns a cost share to each agent and the budget balance condition is $\sum_{i \in N} y_i = C(N,c)$. A cost sharing solution assigns a cost allocation $y(N,c)$ to any admissible mcst problem (N,c).

The Kar solution was explicitly defined and characterized in Kar (2002). It is the Shapley value of the game C. More precisely,

$$y_i^k(N,c) = Sh_i(C) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C(S \cup \{i\}, c) - C(S,c) \right)$$

for all $i \in N$, with $C(\emptyset, c) = 0$.

The core is the set of allocations such that no coalition pays more than its stand-alone cost. Formally, it is the set of allocations y such that for all $S \subseteq N$, $\sum_{i \in S} y_i \leq C(S,c)$. An allocation in the core is said to be stable, as no coalition has an incentive to leave the group and realize the project on its own.

As first noted in Bird (1976), the Kar solution might propose an allocation outside of the core. This comes from the fact the stand-alone cost function is not always concave.[1] Recall that a game C is said to be concave if

$$C(S \cup \{i\}) - C(S) \geq C(T \cup \{i\}) - C(T)$$

for any $S \subset T \subseteq N \setminus \{i\}$.

3 The Kar solution and the core

Consider the three-player example in Figure 1. Agents are identified in the circles, while the cost of each edge appears next to it.

Agent 3's direct connection to the source costs him 1, but with the help of either agent 1 or agent 2, he can connect to the source at no cost. The cooperation gain is one. The only core allocation is $y_1 = y_2 = y_3 = 0$. The Kar solution assigns $\frac{1}{3}$ to agent 3 and -$\frac{1}{6}$ to both agents 1 and 2. Observe that we have free paths $\{(0,1),(1,3)\}$ and $\{(0,2),(2,3)\}$, but that the direct cost to connect 0 and 3 is one.

[1] We are guaranteed that the Shapley value is in the core only for concave games (Shapley (1967)).

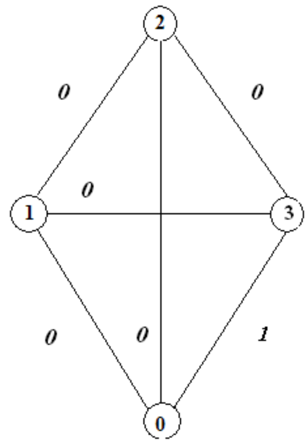

Figure 1: Three-player example

The situations described in Theorem 1 and in the example are exactly as in the glove-market game of Shapley and Shubik (1969). In that game, agents are endowed with exactly one glove, which could be a right-handed or a left-handed glove. A value is created if they can combine a left-handed and a right-handed glove. They can trade freely with each other. If the number of right- and left-handed gloves are not the same, the core predicts that the agents on the "long" side of the market will engage in a Bertrand-like competition that will drive the price to zero, enabling the agents on the "short" side of the market to capture all of the surplus. This result is true even if the quantities are virtually the same, say 1001 right-handed gloves and 1000 left-handed gloves.[2]

While the logic of competition makes sense, it is difficult to qualify this allocation as fair. Even though it takes two agents to generate a gain, one gets nothing while the other pockets all of the surplus. In addition, the core fails to account for the possibility that agents on the "long" side collude to extract some surplus from those on the "short" side. The Shapley value, on the other hand, does leave some surplus to those on the "long" side of the market.

The example in Figure 1 can be expressed in exactly the same manner: The following table gives, for each coalition S, the cost $C(S)$ and the surplus $V(S)$ generated in the private property approach, with $V(S) = \sum_{i \in S} C(\{i\}) - C(S)$.

Table 1: Cost and surplus functions for Example 1.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$C(S)$	0	0	1	0	0	0	0
$V(S)$	0	0	0	0	1	1	1

The surplus is only obtained if we combine agent's 3 willingness to connect to the source with the ability of agent 1 or agent 2 to connect him at no cost. The surplus function V is identical to the glove-market game when agents 1 and 2 have right-handed gloves and agent 3 a left-handed glove. The core predicts that agent 3 will be able to pit agents 1 and 2 against each other, allowing him to extract all of the surplus. The Kar solution still predicts that agent 3 will extract a larger share of the surplus $\left(\frac{2}{3}\right)$ but leaves some part to agents 1 and 2. From a fairness viewpoint, it is difficult to justify the core allocation: agent 3 cannot generate the surplus by himself but is somehow able to extract all of it. From a stability point of view, if we can say that agent 3 can threaten to leave with agent 2 if

[2] A link was previously established between mcst problems and the glove-market example. Following the nomenclature of Apartsin and Holzman (2003), the example is a unitary glove-market game. Kalai and Zemel (1982) show that glove-games in general are equivalent to totally balanced games, of which mcst problems are part of (Bird (1976)).

he does not get all of the surplus, we can also say that agents 1 and 2 can use the threat of kicking agent 3 out if he does not surrender at least part of the surplus to them.

It turns out that all situations where a mcst problem generates a non-concave game is similar to Example 1.

Theorem 1 *A cost matrix c that generates a non-concave stand-alone game $C(N,c)$ is such that there exists $i,j \in N_0$, $S,T \subseteq N \backslash \{i,j\}$ such that $max(c_{0i},c_{0j}) \geq c_{ij} > \bar{c}_{ij}^S, \bar{c}_{ij}^T$.*

The result is a restatement of the main theorem in Trudeau (2012), where it was written in terms of cycles.[3]

For elementary cost matrices, Theorem 1 has the following meaning: there exists a pair of agents (or an agent and the source) that can get a cheaper connection from one to the other from either coalitions S or T, with S and T disjoint. Therefore, we can recover Example 1, with S and T repectively playing the roles of agents 1 and 2 and the coalition $\{i,j\}$ playing the role of agent 3. Since any mcst problem can be expressed as a sum of sub-problems over elementary cost matrices (see for instance the property of Piecewise Linearity in Bogomolnaia and Moulin (2010)), the above interpretation can be extended to any mcst problem.

Note that this discussion only makes sense in the private property approach, where agents are responsible for their location and can be rewarded or punished depending on the value generated by it. If we used the common property approach instead, then in the example agent 3 would have the right to connect to the source through the locations of agents 1 and 2 anyway. However, since the Kar solution only makes sense in the private property approach, the questioning of the fairness of the core allocations is apropos. While stability is a crucial concept, we must understand the limits of the core concept and weigh its benefits against potential clashes with fairness.

4 Conclusion

While the Kar solution is a direct application of the well-studied Shapley value to mcst problems, it has been mostly dismissed in the literature, since it sometimes proposes allocations that are outside of the core. Examining the situations where this can occur shows us that those are conceptually similar to the glove-market games, where core allocations can be viewed as unfair. The use of the private property approach and of the Kar solution supposes that we adhere to the notion of responsibility towards our location. That implies that badly located agents should compensate the agents that help them connect at a cheaper cost, a notion based on the principle of fairness in the division of the cooperation surplus. Therefore, core selection should be seen as incompatible with this notion of responsibility. The focus of the literature on core selection implies, in the private property approach, a precedency of stability over fairness. Since the choice between stability and fairness is not trivial, the Kar solution should be seen as a relevant cost sharing solution.

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[3]The condition $max(c_{0i},c_{0j}) \geq c_{ij}$ guarantees that the edge (i,j) is used by coalition $\{i,j\}$.

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