On Continuous Cost Sharing Solutions for Minimum Cost Spanning Tree Problems

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Abstract

A review of continuous cost sharing solutions for the minimum cost spanning tree problem is proposed, with a particular focus on the folk and Kar solutions. We compare the characterizations proposed, helped by some new results on equivalencies between sets of properties.

1 Introduction

Minimum cost spanning tree (mcst) problems model a situation where agents are located at different points and need to be connected to a source in order to obtain a good or information. Agents do not care if they are connected directly to the source or indirectly through other agents who are. The cost to build a link between two agents is the same whether one or ten agents use a particular link. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks. The model has received considerable attention, no doubt because it is easy to find the optimal network configuration, which is called a mcst, but dividing the cost of the project is not trivial.

Finding the optimal configuration of the network was the focus of the early operations research literature, and it provided efficient algorithms (Boruvka (1926), Kruskal (1956), Prim (1957)). Claus and Kleitman (1973) first discussed the cost sharing problem associated with a mcst problem, and Bird (1976) was the first to study the problem with cooperative game theory tools. Bird provided a method based on Prim's algorithm. That method, now known as the Bird solution, has the advantage of also being in the core, meaning that no group of agents prefers to quit and take on the project alone. Bird also introduced the concept of irreducible cost matrices, where we reduce the cost of each edge as much as possible, with the constraint to leave the total cost of the project unchanged. The irreducible matrix was further studied by Aarts and Driessen (1993). Early work on mcst problems also include studies of the structure of the game, its core and nucleolus by Granot and Huberman (1981, 1982, 1984).

The Bird solution, while interesting, has the unappealing characteristic of depending on the structure of the mcst. Then, small changes in the structure of the problem can lead to drastic changes in the cost shares.[1] The focus of this article is to review the literature on continuous solutions, which have been the subject of a considerable amount of attention over the last dozen years.

It is first worth noting that there are many ways to interpret the mcst problem, with the key debatable point being the property rights of the agents over their location in the network. This can be summed up by the following question: if a coalition S acts on its own and builds its network independently, can it use the locations of agents outside of S to connect to the source? If we allow

[1] Another example of non-continuous cost sharing solutions is the method introduced by Dutta and Kar (2004). Some recent attention, including new characterizations of the Bird solution, can be found in Gómez-Rúa and Vidal-Puga (2011).

players to use other's locations, the corresponding stand-alone cost game is monotonically increasing (it cannot be less expensive to add agents to a coalition) and therefore, cost shares should be non-negative. If we don't allow agents to connect using another's location, then we lose this monotonicity property. The effect of the addition of an agent is twofold: i) we have an extra agent to connect and ii) we have many more network configurations available. This second effect is not present in a game where locations are common property. With it, the total effect on cost is uncertain. An agent who gives access to an efficient network configuration can reduce total cost, thus providing justification for a negative cost share. We call the first approach the *common property* approach, while the second one is the *private property* approach.

The vast majority of the literature insert the mcst problem in the private property approach,[2] although many properties and solutions proposed are closer in spirit to the common property approach. In the private property approach, it is natural to hold agents responsible for the cost of the edges adjacent to their location, rewarding well-situated agents and punishing those that are surrounded by expensive links. The common property approach greatly limits that responsibility.

The Shapley value of the stand-alone cost game associated to the mcst problem under the private property approach was formally introduced and characterized by Kar (2002). The solution is now known as the Kar solution. Fitting completely in the private property approach, it allows negative cost shares to reward some well-located agents, treats asymmetric players differently and is responsive to changes. Its main drawback is that it does not guarantee a core allocation, thus not guaranteeing stability. A popular alternative is what is now known as the folk solution. First appearing in Feltkamp et al. (1994) as the Equal Remaining Obligation solution, then later in Branzei et al. (2004) as the P-value, it is also the average of the family of population monotonic solutions defined in Norde et al. (2004). Bergantiños and Vidal-Puga (2007a) brought much more attention to the solution after introducing it as the Shapley value of the stand-alone game of the cost function defined using the irreducible cost matrix. They also showed the equivalencies between the different approaches used to get to the solution. The folk solution is always a core allocation and is population monotonic. However, by the nature of the irreducible cost matrix, it throws away much information, making it much less responsive than the Kar solution.

The main goal of this article is to present in the same framework the numerous characterizations of the folk solution, as well as the few of the Kar solution, allowing a clear comparison among characterizations of the same solution and making easier to notice the key differences between the Kar and folk solutions. We will also discuss the families of obligation and generalized obligation solutions that include the folk solution. To help see the differences between the various characterizations, a few original results are presented in the form of equivalencies between sets of properties.

The paper is organized as follows. Section 2 formally introduces the model and the two main solutions. In Section 3, the properties used in characterizations are defined and discussed. Relationships between these properties are also presented. Characterizations appear in Section 4, while Section 5 briefly presents some recent developments in the literature.

2 The setting

2.1 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of potential participants and $N \subseteq \mathcal{N}$ be the set of agents that actually need to be connected to the source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with c_e representing the cost of edge e. Let $\Gamma(N)$ be the set of all cost vectors when the set of agents is N, with $N \subseteq \mathcal{N}$. Since c assigns cost to all edges e, we often abuse language and call c a cost matrix. A minimum cost spanning tree

[2] A notable exception is Bogomolnaia and Moulin (2010).

problem is a triple $(0,N,c)$. Since 0 does not change, we omit it in the following and simply identify a mcst problem as (N,c), with $N \subseteq \mathcal{N}$ and $c \in \Gamma(N)$.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_k, i_{k+1}), with $k \in [0, K-1]$ and such that $i_0 = i_K = l$ and $i_1, ..., i_{K-1}$ distinct and different than l. A path p_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k \in [0, K-1]$, containing no cycle and such that $i_0 = l$ and $i_K = m$ and $i_1, ..., i_{K-1}$ distinct and different than l and m.

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges. Its associated cost is $\sum_{e \in t} c_e$.

We briefly present two algorithms to find the minimum cost spanning tree and the associated minimum cost of connecting N to the source. The algorithm introduced by Prim (1957) is as follows. First, pick an edge $(0,i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. In the second step, we choose an edge with the smallest cost connecting an agent in $N \backslash \{i\}$ either directly to the source or to i, which is connected. We continue until all agents are connected, at each step connecting an agent not already connected to an agent already connected or to the source. The algorithm proposed by Kruskal (1956) is similar. First, among all edges, we select the cheapest edge. At each step, we select the cheapest edge among those not previously selected, with the constraint that adding the edge to those chosen in previous steps does not introduce a cycle. We conclude when all agents are connected.

Let $C(N,c)$ be the associated cost. Note that the mcst might not be unique. Let $t^*(c)$ be a mcst and $T^*(c)$ be the set of all mcst for the cost matrix c. Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S,c)$ be the cost of the mcst of the problem (S, c^S). Given these definitions, we say that C is the stand-alone cost function associated with c. Let $C^{CP}(S,c)$ be the cost to connect all agents in S to the source, using all locations in N_0. C^{CP} is the stand-alone cost game generated by the mcst problem (N,c) when we take the common property rights approach. It is easy to see that $C^{CP}(S,c) = \min_{T \supseteq S} C(T,c)$ and $C^{CP}(S,c) \leq C(S,c)$.

2.2 Cost sharing solutions

For a mcst problem (N,c), a cost allocation $y \in \mathbb{R}^N$ assigns a cost share to each agent, and the budget balance condition is $\sum_{i \in N} y_i = C(N,c)$. A cost sharing solution (or rule) assigns a cost allocation $y(N,c)$ to any mcst problem (N,c). We introduce the two solutions that are the focus of the paper.

The Kar solution was explicitly defined and characterized in Kar (2002). It is the Shapley value of the game C. More precisely,

$$y_i^k(N,c) = Sh_i(C) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \left(C(S \cup \{i\}, c) - C(S,c)\right)$$

for all $i \in N$, with $C(\emptyset, c) = 0$.

As mentioned in the introduction, the so-called folk solution has been obtained in different ways. We focus on the approach of Bergantiños and Vidal-Puga (2007a), which uses the Shapley value, thus allowing a clear comparison with the Kar solution.

From any cost matrix c, randomly select a mcst $t^*(c) \in T^*(c)$ and let $p_{ij}^{t^*}$ be the path between i and j along the mcst $t^*(c)$. We define the irreducible cost matrix c^* as follows:

$$c_{ij}^* = \max_{e \in p_{ij}^{t^*}} c_e.$$

The folk solution is the Shapley value of the stand-alone cost function associated to c^*, defined as $C^*(S,c) = C(S,c^*)$ for all $S \subseteq N$.

Bogomolnaia and Moulin (2010) offer a closed-form expression of the folk solution. Fix i and rearrange the cost c_{ij}^* of the $n-1$ edges connecting agent i to other agents in increasing order as c_i^{*k}

such that $c_i^{*1} \leq c_i^{*2} \leq ... \leq c_i^{*(n-1)}$. Then, the folk solution $y^f(N,c)$ can be written as

$$y_i^f(N,c) = \frac{1}{n}c_{0i}^* + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\left\{c_i^{*k}, c_{0i}^*\right\}.$$

Another interpretation, found in Bergantiños and Vidal-Puga (2007b), uses the notion of an optimistic game. This game assigns to any coalition S the cost of connecting its members to the source under the assumption that agents in $N \backslash S$ are already connected and that locations are common property. We can then define the folk solution as the Shapley value of the corresponding stand-alone game.

By contrast, the Kar solution is the Shapley value of the stand-alone game where a coalition assumes that others are not connected and that locations are private property, also called the pessimistic stand-alone game.

3 Properties

We examine the properties used to describe and characterize the various cost sharing solutions. Since the main contribution is to combine the various properties, we refer to the original articles for the origins of these properties and links with the literature to focus on differences and interpretations of these properties. We divide the properties in three groups: stability properties, properties that try to simplify the problem and properties based on comparisons between agents or between problems.

3.1 Stability properties

In many contexts, there is a concern that the agents will not freely agree to cooperate. The classic Core Selection property makes sure that no group of agents is assigned more than the cost to realize the project by themselves.

Core Selection (CS): For any mcst problem (N,c) and any $S \subseteq N$, $\sum_{i \in S} y_i(N,c) \leq C(S,c)$.

Population Monotonicity ensures that no agent will oppose the addition of new members to the group.

Population Monotonicity (PM): For any mcst problem (N,c) and any $i \in S \subseteq N$, $y_i(N,c) \leq y_i(S,c^S)$.

3.2 Simplification properties

There are many reasons to want to simplify cost sharing problems. Often, the division of the problem into two parts is natural, for instance because two groups of agents would not gain if they cooperated with one another or because the construction of the mcst is done in two separate steps. We may also want to throw away some of the information contained in the cost matrix. The first two properties allow us to split the problem into a sum of smaller, simpler problems, and, they are closely related to the classic Additivity property found in traditional cost sharing models. If for two cost matrices c, c' there is a common ranking of the edges from cheapest to most expensive, then the cost shares on $c+c'$ are the sum of the cost shares on c and c' according to Piecewise Linearity. Restricted Additivity is stronger, as it only requires that c and c' share a common mcst t, for which there is a common ranking of edges on this t, from cheapest to most expensive.

Piecewise Linearity (PL)[3]: For any mcst problems (N,c) and (N,c'), if there exists an order of the edges $\sigma : N_0^p \to \left\{1, ..., \frac{n(n+1)}{2}\right\}$ such that for any $e, e' \in N_0^p$, if $\sigma(e) \leq \sigma(e')$, we have $c_e \leq c_{e'}$ and $c'_e \leq c'_{e'}$, then, $y(N, c+c') = y(N,c) + y(N,c')$.

Restricted Additivity (RA)[4]: For any mcst problems (N,c) and (N,c'), if there exists $t \in$

[3] Piecewise Linearity is labelled Cone-wise positive linearity in Bergantinos and Vidal-Puga (2009a) and Bergantinos and Kar (2010).

[4] Restricted Addivity is called Constrained Additivity in Lorenzo and Lorenzo-Freire (2009).

$T^*(c) \cap T^*(c')$ and an order of its edges $\pi : t \to \{1, ..., |N|\}$ such that for any $e, e' \in t$, if $\pi(e) \leq \pi(e')$, we have $c_e \leq c_{e'}$ and $c'_e \leq c'_{e'}$, then, $y(N, c + c') = y(N, c) + y(N, c')$.

It is easy to see that Restricted Additivity implies Piecewise Linearity. We next consider a basic property that says that if an edge is not used by any coalition, then raising the cost of that edge should have no impact on cost shares. We need the following definitions.

An edge (i, j) is irrelevant if $c_{ij} > \max[c_{0i}, c_{0j}]$. Such an edge is never used, as it is always preferable to connect agents i and j through the source.

Independence of Irrelevant Edges (IIE): For any mcst problems (N, c) and (N, c'), if $\max[c_{0i}, c_{0j}] \leq c_{ij} < c'_{ij}$ and $c_e = c'_e$ else, then $y(N, c) = y(N, c')$.

Independence of Irrelevant Edges eliminates the real cost of irrelevant edges, that are not used by any coalition. Reductionism goes much further, requiring that the cost shares depend only on the irreducible matrix and thus only on the cost of edges that are part of a mcst. Therefore, much of the information contained in the cost matrix is discarded, as the irreducible matrix contains at most $|N|$ different values for the cost of edges. On the bright side, it makes the problem simpler and eases computation.

Reductionism (RED)[5]: For any mcst problem (N, c), $y(N, c) = y(N, c^*)$.

The next two properties deal with cases where S and $N \backslash S$ connect independently to the source. The idea is to allow one to compute the shares separately on the problems restricted to S and $N \backslash S$ respectively. Separability allows this as soon as the (stand-alone) costs to connect S and $N \backslash S$ to the source sum to the total cost of the project. Then, there is no gain for S to cooperate with $N \backslash S$. Group Independence applies less often, as it requires that S and $N \backslash S$ be completely independent: no group in S has any gain if it cooperates with any group in $N \backslash S$.

Group Independence (GI): For any mcst problem (N, c), if $S \subset N$ is such that for all $i \in S$ and $j \in N \backslash S$, $c_{ij} \geq \max[c_{0i}, c_{0j}]$, then, $y_i(N, c) = \begin{cases} y_i(S, c^S) \text{ if } i \in S \\ y_i(N \backslash S,\ c^{N \backslash S}) \text{ if } i \in N \backslash S \end{cases}$.

Separability (SEP): For any mcst problem (N, c), if $C(S, c) + C(N \backslash S, c) = C(N, c)$, then, $y_i(N, c) = \begin{cases} y_i(S, c^S) \text{ if } i \in S \\ y_i(N \backslash S,\ c^{N \backslash S}) \text{ if } i \in N \backslash S \end{cases}$.

It is clear that Separability implies Group Independence. The next two properties are based on the observation that if a cost matrix has no irrelevant edges, then there is always a mcst where only one agent is directly connected to the source. The problem of finding a mcst can then be divided in two: finding who to connect to the source and connecting all agents together. Problem Separation says that we should be able to divide the cost sharing problem in the same manner: first by sharing the cost of the source connection problem, then by sharing the cost of the agent connection problem. We need the following definitions:

Let $\hat{c}$ be the source connection problem associated with c : for all $i \in N$, $\hat{c}_{0i} = c_{0i}$, while $\hat{c}_{ij} = 0$ for all $i, j \in N$. Then, all that is left are the costs to connect agents to the source. The mcst is such that one agent is connected to the source (i such that $\hat{c}_{0i} \leq \hat{c}_{0j}$ for all $j \in N$) and all others are connected to him (at no cost since $\hat{c}_{ij} = 0$ for all $i, j \in N$). Let $\tilde{c}$ be the agent connection problem associated with c : for all $i, j \in N$, $\tilde{c}_{ij} = c_{ij}$, while $\tilde{c}_{0i} = \max_{e \in N_0^p} c_e$ for all $i \in N$. Then, all agents have the same (high) cost to connect to the source, so the mcst is such that only one (random) agent is connected to the source, and all other agents are connected through this agent. Let $\dot{c}$ be defined as follows: $\dot{c}_{0i} = \max_{e \in N_0^p} c_e$ for all $i \in N$ and $\dot{c}_e = 0$ otherwise.

We can show that for any $S \subseteq N$, $C(S, c) = C(S, \hat{c}) + C(S, \tilde{c}) - C(S, \dot{c})$. Note that since we added the extra cost $\max_{e \in N_0^p} c_e$ to the agent connection problem, we remove it with $C(S, \dot{c})$.

Problem Separation (PS): For any mcst problem (N, c) such that c contains no irrelevant edge, $y_i(N, c) = y_i(N, \hat{c}) + y_i(N, \tilde{c}) - y_i(N, \dot{c})$ for all $i \in N$.[6]

Weak Problem Separation applies the property only in situations for which the most expensive edge

[5] Reductionism is called Independence of Irrelevant Trees in Bergantinos and Vidal-Puga (2007a).

[6] If y satisfies Symmetry, then $y_i(N, \dot{c}) = \frac{1}{|N|} \max_{e \in N_0^P} c_e$ for all $i \in N$. The extra cost added to the agent connection problem is split evenly among agents.

of a mcst is the edge connecting an agent to the source. In those situations, it is clear that we only want to connect one agent to the source, lending more credibility to the idea of splitting a problem in its source and agent connection subproblems.

Weak Problem Separation (WPS): For any mcst problem (N,c) such that c contains no irrelevant edge, if $c_e \leq \min_{i\in N} c_{0i}$ for all $e \in t^*(c)$, all $t^* \in T^*(c)$, then $y_i(N,c) = y_i(N,\hat{c}) + y_i(N,\tilde{c}) - y_i(N,\dot{c})$ for all $i \in N$.

Among these properties, only Independence of Irrelevant Edges and Group Separability seem to be universally desirable. While others do simplify the problem, it is not always obvious that it is appropriate to use them.

3.3 Comparison properties

Cost sharing solutions are often evaluated on how they compare agents between them or how the shares change when there are small modifications to the problem. We start with the unavoidable property of Symmetry, which is such that if two agents have the same characteristics, which in our case are the cost of edges to other agents, then they should be treated equally.

Symmetry (SYM)[7]: For any mcst problem (N,c), if i,j are such that $c_{ik} = c_{jk}$ for all $k \in N_0\backslash\{i,j\}$, then $y_i(N,c) = y_j(N,c)$.

The Anonymity property is slightly stronger: it says that the costs allocated to the agents should not depend on their name. We need the following notation: A permutation π of $N = \{1,..,n\}$ is a one-to-one mapping from N to N. π_i is the new rank of agent i. For any $c \in \Gamma(N)$, $\pi c \in \Gamma(N)$ is such that $(\pi c)_{\pi_i\pi_j} = c_{ij}$ and $(\pi c)_{0\pi_k} = c_{0k}$ for all $i,j,k \in N$. For any $y \in \mathbb{R}^N_+$, πy is such that $\pi y_{\pi_i} = y_i$ for all $i \in N$.

Anonymity (A): For any mcst problem (N,c) and permutation π of N, we have $\pi y(N,c) = y(N,\pi c)$.

Ranking goes further: if the location of agent i is no worse than the location of agent j, then i should not pay more than j. Strict Ranking goes beyond that: if that location is strictly better, then the allocation of i should be strictly less than the allocation of j.[8]

Ranking (R): For any mcst problem (N,c), if $c_{ik} \leq c_{jk}$ for all $k \in N_0\backslash\{i,j\}$, then $y_i(N,c) \leq y_j(N,c)$.

Strict Ranking (SR): For any mcst problem (N,c), if $c_{ik} \leq c_{jk}$ for all $k \in N_0\backslash\{i,j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0\backslash\{i,j\}$, with $c_{il} < \max[c_{0i}, c_{0l}]$, then $y_i(N,c) < y_j(N,c)$.

The next five properties concern the modifications to cost shares that occur when we alter the cost of one or more edges in the cost matrix. Cost Monotonicity is a minimal property that says that if the cost of the edge between i and j decreases, then i and j should not suffer. If that edge is used by at least one coalition, Strict Cost Monotonicity imposes that the allocations of i and j strictly decrease.

Cost Monotonicity (CM): For any mcst problems (N,c) and (N,c') and $i,j \in N_0$, if $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(N,c') \leq y_k(N,c)$ for $k \in \{i,j\}\setminus\{0\}$.

Strict Cost Monotonicity (SCM): For any mcst problems (N,c) and (N,c') and $i,j \in N_0$, if $c_{ij} \leq \max_{k\in\{i,j\}\setminus\{0\}} c_{0k}$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(N,c') < y_k(N,c)$ for $k \in \{i,j\}\setminus\{0\}$.

Strict Cost Monotonicity conveys the idea that a pair of agents is responsible for the cost of the edge connecting them. Equal Treatment goes further: there should be equal responsibility. If the cost of the edge goes down, both agents should see their cost shares change by the same amount. Weak Equal Treatment applies the same idea, but only to changes that do not affect the total cost of the project (changes on edges that are not part of any mcst).

Equal Treatment (ET): For any mcst problems (N,c) and (N,c') and $i,j \in N$ such that $c_{ij} > c'_{ij}$ and $c_e = c'_e$ else, we have $y_i(N,c) - y_i(N,c') = y_j(N,c) - y_j(N,c')$.

[7] The property is labelled Equal Treatment of Equals in Bergantinos and Kar (2010).

[8] The additional constraint in the property is to make sure that at least one of the strictly better edge is not an irrelevant edge.

Weak Equal Treatment (WET): For any mcst problems (N,c) and (N,c') and $i,j\in N$ such that $c_{ij}>c'_{ij}$ and $c_e=c'_e$ else, with $C(N,c)=C(N,c')$, we have $y_i(N,c)-y_i(N,c')=y_j(N,c)-y_j(N,c')$.

The following two properties go in a completely different direction and fit much better with the common property approach. Solidarity says that if the cost of an edge decreases, then nobody's cost should increase. It is similar to Cost Monotonicity, but not just limited to the agents adjacent to that edge. Weak Solidarity limits that property to edges that are part of a mcst and therefore have an impact on total cost.

Solidarity (SOL)[9]**:** For any mcst problems (N,c) and (N,c') if $c_{ij}<c'_{ij}$ and $c_e=c'_e$ else, then $y_k(N,c)\leq y_k(N,c')$ for all $k\in N$.

Weak Solidarity (WSOL): For any mcst problems (N,c) and (N,c'), if $c_{ij}<c'_{ij}$ and $c_e=c'_e$ else and $C(N,c)<C(N,c')$, then $y_k(N,c)\leq y_k(N,c')$ for all $k\in N$.

Suppose that the costs to connect agents to the source are identical and more expensive than the costs to connect agents among themselves. Then, Equal Share of Extra Cost says that if that identical cost to connect agents to the source increases, then the extra cost should be split equally among agents.

Equal Share of Extra Cost (ESEC): For any mcst problems (N,c) and (N,c') if $c_{0i}=c_0$, $c'_{0i}=c'_0$ for all $i\in N$, $c'_0>c_0\geq 0$ and $c_{jk}=c'_{jk}\leq c_0$ for all $j,k\in N$, then $y_i(N,c')=y_i(N,c)+\frac{c'_0-c_0}{|N|}$.

Among these properties, it would be very difficult to argue against Symmetry, Ranking, Cost Monotonicity and Equal Share of Extra Cost. Strict Ranking is also highly desirable because, if it is not satisfied, we will treat some asymmetric players symetrically. Equal Treatment and Solidarity are dependent on the interpretation we have of the problem and on property rights.

3.4 Relations between properties

These properties are not completely independent. In addition to the obvious relationships between strong and weak versions of the same properties, we have the following implications.

Lemma 1 *i) Population Monotonicity implies Core Selection and Separability.*
ii) Solidarity implies Reductionism and Cost Monotonicity.
iii) Reductionism implies Weak Equal Treatment.

Parts i) and ii) were shown in Bergantiños and Vidal-Puga (2007a), part iii) in Trudeau (2010).

We have seen that Restricted Additivity and Separability imply, respectively, Piecewise Linearity and Group Independence. They also convey the additional idea of reliance on the irreducible cost matrix. This intuition proves to be true, as shown in the following lemma.

Lemma 2 *If N contains 3 or more agents a solution y satisfies Restricted Additivity and Separability if and only if it satisfies Piecewise Linearity, Group Independence and Reductionism.*

Proof. We first show that Restricted Additivity and Separability imply Piecewise Linearity, Reductionism and Group Independence. It is clear that Restricted Additivity implies Piecewise Linearity and that Separability implies Group Independence. We show that Restricted Additivity and Separability imply Reductionism. Take an arbitrary cost matrix c and define c' such that $c'_e=c_e-c^*_e$ for all edges e. By definition, c and c^* have an identical mcst t (same edges, same costs) and $c'_e=0$ for all $e\in t$. Therefore, we can apply Restricted Additivity, such that $y(N,c)=y(N,c^*)+y(N,c')$. By Restricted Additivity, we can write $y(N,c')=\sum_{(k,l)\in N_0^p}c'_{kl}y(N,c^{kl})$ where $c^{kl}_e=1$ if $e=(k,l)$ and 0 otherwise. If $0\notin\{k,l\}$, then, we have that $\sum_{i\in N}C(\{i\},c^{kl})=C(N,c^{kl})=0$. By Separability, $y_i(N,c^{kl})=y_i(\{i\},(c^{kl})^{\{i\}})=0$ for all $i\in N$. If $\{k,l\}=\{0,j\}$, then for each $i\in N\backslash\{j\}$, we have $C(N\backslash\{i\},c^{kl})=C(\{i\},c^{kl})=0$ (as we have three or more players). By Separability,

[9] Solidarity is called Strong Cost Monotonicity in Bergantiños and Kar (2010), Bergantiños et al. (2010) and Bergantiños et al. (2011).

$y_i(N, c^{kl}) = y_i(\{i\}, (c^{kl})^{\{i\}}) = 0$ for all $i \in N\backslash\{j\}$. By budget-balance, $y_j(N, c^{kl}) = 0$. Therefore, $y(N, c) = y(N, c^*)$.

Next, we show that Piecewise Linearity, Reductionism and Group Independence imply Restricted Additivity and Separability. We first show Restricted Additivity. Suppose that c and c' have a common mcst t for which the edges are ranked in the same manner from cheapest to most expensive. For any $i, j \in N_0$, let p^t_{ij} be the path between i and j on the mcst t. By definition, $c^*_{ij} = \min_{e \in p^t_{ij}} c_e$ and $c^{*\prime}_{ij} = \min_{e \in p^t_{ij}} c'_e$. Let $d = c + c'$. Clearly, $d^*_{ij} = c^*_{ij} + c^{*\prime}_{ij}$. By Reductionism, $y(N, d) = y(N, d^*)$. By Piecewise Linearity, $y(N, d^*) = y(N, c^*) + y(N, c^{*\prime})$. By Reductionism, $y(N, c^*) + y(N, c^{*\prime}) = y(N, c) + y(N, c')$. Restricted Additivity is satisfied.

To show Separability, we show that under Reductionism, Group Independence is equivalent to Separability. Suppose that, for an irreducible matrix c^*, we have $C(S, c^*) + C(N\backslash S, c^*) = C(N, c^*)$, so that we could apply Separability on S and $N\backslash S$. We can apply Group Independence only if $C(R, c^*) + C(T, c^*) = C(R \cup T, c^*)$ for all $R \subseteq S$ and $T \subseteq N\backslash S$. Since $C(S, c^*) + C(N\backslash S, c^*) = C(N, c^*)$, we have a mcst t^S for c^* that connects agents in S to the source using only the location of agents in S. We have a similar mcst $t^{N\backslash S}$ for $N\backslash S$. Let $t = t^S \cup t^{N\backslash S}$. By the property of irreducible matrices, for any $i \in S$ and $j \in N\backslash S$,

$$
\begin{aligned}
c^*_{ij} &= \max_{e \in p^t_{ij}} c^*_e \\
&= \max\left(\max_{e \in p^{t^S}_{0i}} c^*_e, \max_{e \in p^{t^{N\backslash S}}_{0j}} c^*_e\right).
\end{aligned}
$$

Therefore, we have that $c^*_{ij} \geq \max_{e \in p^{t^S}_{0i}} c^*_e = c^*_{0i}$ and $c^*_{ij} \geq \max_{e \in p^{t^{N\backslash S}}_{0j}} \bar{c}_e = \bar{c}_{0j}$. Therefore, there is never any gain for a member of S to connect with a member of $N\backslash S$. With an irreducible matrix, in all situations where we can apply Separability, we can also apply Group Independence. ■

Therefore, even though Restricted Additivity and Separability seem completely reasonable at first glance, it is important to see that they imply Reductionism in addition to their respective weaker versions, Piecewise Linearity and Group Independence. This result eases the comparison between the various characterizations proposed in the literature. For the same reason, we show that we can also decompose the Solidarity property.

Lemma 3 *A solution y satisfies Solidarity if and only it satisfies Weak Solidarity and Reductionism.*

The result can be easily seen as Reductionism is the consequence of Solidarity when applied to an edge that is not part of any mcst.

We conclude this section by showing some impossibility results.

Lemma 4 *i) No solution satisfies Strict Ranking and Core Selection.*
ii) No solution satisfies Strict Cost Monotonicity and Core Selection.
iii) No solution satisfies Strict Ranking and Reductionism.
iv) No solution satisfies Strict Cost Monotonicity and Reductionism.

Parts i) and ii) can be found in Trudeau (2011a) while parts iii) and iv) are weaker versions of the results of Bogomolnaia and Moulin (2010). Thus, Strict Ranking and Strict Cost Monotonicity, which are at the heart of the private property approach, are incompatible with Core Selection. By Lemma 1, it also means that they are incompatible with Population Monotonicity. Parts iii) and iv) show that any solution that satisfies Reductionism is much closer to the common than the private property approach.

4 Characterizations

We are now ready to examine the characterizations offered in the literature for the folk solution, the obligation solutions and the Kar solution.

4.1 Folk solution

As mentioned, the folk solution has received the most attention, and it is not surprising that it also has the most characterizations. Using the results of the previous section, we are able to present them in a compact manner.

Theorem 5 *Suppose that N contains three or more agents. A solution y is the folk solution if and only if it satisfies:*

i) Equal Share of Extra Cost, Reductionism and Group Independence. (Adapted from Bergantiños and Vidal-Puga (2007a))

ii) Piecewise Linearity, Symmetry, Reductionism and Core Selection. (Bogomolnaia and Moulin (2010), Branzei et al. (2004))

iii) Piecewise Linearity, Symmetry, Reductionism and Group Independence. (Adapted from Bergantiños and Vidal-Puga (2009a))

Proof. ii) is reproduced exactly. i) was proven with Separability instead of Group Independence. The proof carries easily with Group Independence. iii) uses Lemma 2 to replace Restricted Additivity and Separability by Piecewise Linearity, Reductionism and Group Independence. ■

As we can see, all characterizations include Reductionism, a disputable property that is obviously close to the spirit of the folk solution as it imposes a reliance on the irreducible matrix. It is easy to see that Piecewise Linearity and Symmetry imply Equal Share of Extra Cost. Therefore, i) is a tighter characterization than iii). While it is more difficult to argue against the principle of Equal Share of Extra Cost than Piecewise Linearity, the decomposition in elementary cost matrices that comes with Piecewise Linearity is instructive. From a normative point of view, i) seems like the most appealing characterization.

Bergantiños and Vidal-Puga (2007a) present a variant of i) using Weak Solidarity and Population Monotonicity instead of Group Independence. Since Group independence is implied by Population Monotonicity, i) is a tighter characterization.

4.2 Obligation solutions

Obligation solutions are an interesting family of cost sharing solutions that include the folk solution. It was introduced in Tijs et al. (2006). The formal definition is involved and we refer to Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010) for it. We focus on the intuition. The solutions are closely related to Kruskal's algorithm: at each step, the cost of the edge that is added to the mcst is paid by the agents who benefit from it. Cost shares are based on an obligation function that specifies the responsibility of individuals towards groups. Bergantiños and Kar (2010) provide a different approach: obligation solutions are marginalistic values of the irreducible cost game. We examine the various characterizations of the family.

Theorem 6 *A solution y is an obligation solution if and only if it satisfies:*

i) Restricted Additivity and Population Monotonicity. (Lorenzo and Lorenzo-Freire (2009))

ii) Restricted Additivity, Reductionism, Weak Solidarity and Core Selection. (Bergantiños et al. (2011))

iii) Restricted Additivity, Reductionism, Weak Solidarity and Separability. (Bergantiños et al. (2011))

iv) Piecewise Linearity, Reductionism, Weak Solidarity and Population Monotonicity.[10] *(Bergantiños and Kar (2010))*

[10] Bergantiños and Kar (2010) also offer an alternative characterization using a property very close to Equal Share of Extra Cost that says that if there are two situations like in the condition for Equal Share of Extra Cost, that extra cost will be shared identicially in both problems.

Proof. i) is reproduced exactly. For the other three characterizations, Solidarity has been replaced by Reductionism and Weak Solidarity. ■

Since Restricted Additivity and Separability are equivalent to Piecewise Linearity, Reductionism and Group Independence, we see that iii) is not quite tight. It can be written as Piecewise Linearity, Reductionism, Weak Solidarity and Group Independence. Since Population Monotonicity implies Separability, iii) is tighter than iv). Bergantiños and Kar (2010) and Bergantiños et al. (2011) show that by adding Symmetry to their characterization we obtain the folk solution. An interesting open question is whether we obtain obligation rules if we remove Symmetry from the characterizations ii) and iii) of Theorem 5.

Bergantinos, Lorenzo and Lorenzo-Freire (2010, 2011) introduce the family of generalized obligation solutions. For an obligation solution, the obligation of an agent depends only on the other agents to which she is connected at each step in Kruskal's algorithm. By contrast, for a generalized obligation solution, the obligation of an agent depends on the whole set of agents and not just on the agents she is connected to.

Theorem 7 *A solution y is a generalized obligation solution if and only if it satisfies Restricted Additivity, Reductionism and Weak Solidarity. (Bergantiños et al. (2010))*

Therefore, we obtain generalized obligation solutions with Restricted Additivity, Reductionism and Weak Solidarity. Adding Core Selection or Separability restricts the set to obligation solutions. Adding Symmetry on top of that leaves us with only the folk solution.

4.3 Kar solution

By contrast to the folk solution, the Kar solution has received little attention. As discussed in the introduction, it has both weaknesses and strengths. In particular, it is much better suited for the private property approach and does not throw away information like the folk solution. We have this characterization for the Kar solution.

Theorem 8 *A solution y is the Kar solution if and only if it satisfies Equal Treatment and Group Independence. (Kar (2002))*[11]

Equal Treatment is a very strong property that not only conveys the idea that agents are responsible for the edges adjacent to their location, but also that agents i and j are equally responsible for the edge connecting them. Comparing this with characterization i) of the folk solution of Theorem 5, we see that Equal Treatment replaces Equal Share of Extra Cost and Reductionism. Since Extra Share of Extra Cost is also satisfied by the Kar solution, a characterization using it together with Group Independence would allow a clear comparison with the folk solution.

4.4 Comparing the Kar and folk solutions

A comparison was attempted in Trudeau (2010), who characterizes a family of solutions that turn out to be the affine combination of the Kar and folk solutions. They are characterized by six relatively mild properties.

Theorem 9 *A solution y satisfies Piecewise Linearity, Anonymity, Independence of Irrelevant Agents, Weak Equal Treatment, Weak Problem Separation and Group Independence if and only if $y = ay^k + (1-a)y^f$, with $a \in \mathbb{R}$. (Trudeau (2010))*

Among those, the Kar solution is the only one that satisfies Problem Separation. The folk solution is the only one that satisfies Core Selection or Population Monotonicity. Also, all members of the family

[11] Kar (2002) actually uses Absence of Cross Subsidization and a weak form of Group Independence. It is easy to check that the combination of these properties is equivalent to our version of Group Independence.

except the folk solution satisfy Strict Ranking and Strict Cost Monotonicity. The characterizations of the solutions have the advantage of using properties that are not associated with one approach or the other. For instance, the folk solution is characterized without Reductionism.

This also clearly establishes the clash between the approaches behind the two solutions. The folk solution deals very well with stability issues but does poorly in terms of responsiveness to changes and responsibility for the adjacent edges. It is the opposite for the Kar solution. To confirm this, we examine which properties are satisfied by the two solutions, with "+" indictating that teh property is satisfied and "-" that it is not.

Table 1: Properties of the folk and Kar solutions

	Stability			Simplification						
	CS	PM	PL	RA	IIE	RED	GI	SEP	PS	WPS
Folk	+	+	+	+	+	+	+	+	-	+
Kar	-	-	+	-	+	-	+	-	+	+

	Comparison										
	SYM	A	R	SR	CM	SCM	SOL	WSOL	ET	WET	ESEC
Folk	+	+	+	-	+	-	+	+	-	+	+
Kar	+	+	+	+	+	+	-	-	+	+	+

Most of these results come from Bergantiños and Vidal-Puga (2007a) and Bergantiños and Vidal-Puga (2008). Strong Core Selection was proven in Bogomolnaia and Moulin (2010), Strict Ranking and Strict Cost Monotonicity in Trudeau (2011a)), Weak Equal Treatment, Piecewise Linearity, Independence of Irrelevant Edges, Group Independence, Problem Separation and Weak Problem Separation in Trudeau (2010). It was first shown in Bergantiños and Vidal-Puga (2009a) that the folk solution satisfies Restricted Additivity. That it is not satisfied by the Kar solution is a new, albeit trivial, result.

We see the clear difference in terms of stability properties. Among comparison properties, the only properties that are not satisfied by both are those that are built on a particular approach. As for simplification properties, we see that the folk solution can be simplified using the irreducible matrix, while the Kar solution can always be divided into the source and agent connection problems. As we've seen in Lemma 2, Reductionism is closely related to Restricted Additivity and Separability.

5 New directions

We conclude this survey by examining recent developments and the directions in which the literature is heading.

The divide between the Kar and folk solutions was further examined in Trudeau (2011a). A new solution is proposed. It still transforms the cost matrix c into a reduced matrix, but one in which the modifications are not as important as in the irreducible matrix. To find the irreducible matrix, we look at paths between i and j, finding the one for which the most expensive edge is as cheap as possible and assigning that cost to edge (i, j) in the irreducible matrix. Instead, we define the cycle-complete matrix, where we look at cycles that go through i and j, finding the one for which the most expensive edge is as cheap as possible and assigning that cost to edge (i, j) in the cycle-complete matrix. The resulting stand-alone game is concave and, therefore, the cycle-complete solution, which is the Shapley value of that game, satisfies Core Selection. While it still fails Population Monotonicity, it now satisfies watered-down versions of Strict Ranking and Strict Monotonicity that are not satisfied by the folk solution, thus allowing for more responsiveness but preserving stability. A formal characterization has not yet been obtained. It does show that Reductionism is not necessary to obtain Core Selection.

In Trudeau (2011b), it is argued that the divide between the Kar and folk solutions might be deeper than it looks. It is shown that the situations where the Kar solution does not provide an allocation in the core are very familiar to the well-studied glove-market games (Shapley and Shubik (1969)), where the core allocates all of the surplus to the side of the market (buyers or sellers) that has fewest members.

The same happens with mcst problems when i and j can use many disjoint coalitions to improve on the cost of their connection to each other. The fairness of these allocations can be questioned. It also does not allow providers of these cheap connections the opportunity to collude to extract some surplus.

Non-cooperative implementation has received some attention. Bergantiños and Vidal-Puga (2010) design non-cooperative games that allow one to obtain the folk or Kar solution as the allocation in equilibrium. Hougaard and Tvede (2011) examine the problem with asymmetric information between the agents and the central planner and focus on truth-telling issues.

Hougaard et al. (2010) examine decentralized pricing for mcst problems, where the goal is to define the share of an agent as a function of only her adjacent nodes. It is then impossible to collect exactly the cost of the project, but they operate under the constraint to amass enough to cover the total cost. They define a pricing scheme that is close to the folk solution and significantly improves on a naive pricing scheme where we would make each agent pay for his stand-alone cost.

In a recent paper, Dutta and Mishra (2011) extend the mcst framework to allow for asymmetric costs, depending if the flow is from i to j or j to i. They provide, for such minimum cost arborescences, a solution that is similar in spirit to the folk solution and show that it satisfies Core Selection. Other recent studies of related network problems include cost spanning trees with groups (Bergantinos and Gomez-Rua (2010, 2011), a capacity model solved by a maximum cost spanning tree (Bogomolnaia et al. (2010)), problems with uncertainties (Moretti et al. (2011)) and the PERT scheduling problem (Bergantiños and Vidal-Puga (2009b)). Extensions to network flow problems (Quant et al. (2006), Trudeau (2009)) might prove to be more difficult as there are no algorithms to solve the problem exactly.

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