A new stable and more responsive solution for mcst problems*

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Abstract

Minimum cost spanning tree (mcst) problems try to connect agents efficiently to a source when agents are located at different points in space and the cost of using an edge is fixed. Two solution concepts to share the common cost of connection among users are based on the Shapley value: the Kar and the folk solutions. The Kar solution applies the Shapley value to the stand-alone cost game, which might be non-concave and non-monotonic, yielding cost allocations that might be unstable or negative. The folk solution modifies the costs of some edges, yielding a monotonic and concave cost game, and thus non-negative allocations that are always in the core.

We show necessary conditions for a mcst problem to generate a non-concave cost game. Using this result, we offer a new solution by modifying the cost of some edges, less than for the folk solution but enough to generate a concave but not necessarily monotonic cost game. Taking the Shapley value of this game gives allocations that are always in the core but possibly negative. It is also a compromise between the Kar and folk solutions as it satisfies more properties that make agents responsible for their locations in the network. We also examine more closely the incompatibilities between properties assuring stability and responsibility for one's location, as well as the link between these properties and the non-negativity of the cost shares.

1 Introduction

Minimum cost spanning tree (mcst) problems model a situation where agents are located at different points and need to be connected to a source in order to obtain a good or information. Agents do not care if they are connected directly to the source or indirectly through other agents who are. The cost to build a link between two agents or an agent and the source is a fixed number, meaning that the cost is the same whether one or ten agents use a particular link. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks.

We are interested in the cost sharing problem related to mcst problems. Once agents decide to do the project and build the network, the common cost of construction must be split among the participants. There has been two approaches to this problem, the dividing point being the property rights of an agent on its location. This can be summed up by the following question: if a coalition S acts on its own and builds its network independently, can it use the locations of agents outside of S to connect to the source? Clearly, the answer changes the game drastically. If we allow players to use other's locations, the corresponsing stand-alone cost game is monotonically increasing (it cannot be less expensive to add agents to a coalition) and therefore, cost shares should be non-negative. If we don't allow agents to connect using other's location, then we lose this monotonicity property. The effect of the addition of an agent is twofold: i) we have an extra agent to connect and ii) we have many

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more network configurations available. This second effect is not present in a game where we allow agents to connect using all locations. With it, the total effect on cost is uncertain. An agent that gives access to an efficient network configuration can therefore reduce total cost, providing justification for a negative cost share.

We call the stand-alone cost game with property rights C, while C^{NR} is the stand-alone cost game with no property rights. It is easy to see that for all coalition S, $C^{NR}(S) \leq C(S)$. Core allocations are such that no subset of agents have incentives to secede from the group to do the project on their own. It is clear that $core(C^{NR}) \subseteq core(C)$.

Distinctions between the two approaches are not always clear in the literature, although most model the game with property rights, but sometimes use concepts more appropriate for the no rights problems. Bird (1976) showed that there always exists at least one allocation in $core(C^{NR})$ and thus in $core(C)$. He showed however that the Shapley value of both games was not necessarily in the core. Granot and Huberman (1982) show that C is not necessarily concave. The notion of concavity is well used in the cost sharing literature and means that if $S \subset T$, the cost to add an agent to coalition T cannot be higher than the cost to add this same agent to S. Concavity induces a very nice structure to the game. In particular, if we rank agents in a particular order and assign to each agent its incremental cost in that order, the resulting allocation is an extreme point of the core, the set of stable allocations (Shapley (1971)). In addition, all extreme points of the core are incremental cost allocations (Ichiishi (1981)). Granot and Huberman (1982) show instead that C is permutationally concave, a weakening of the concept that says that incremental costs are (weakly) decreasing only with respect to a particular ordering of the agents. By contrast, concave games have decreasing incremental costs with respect to all orderings of the agents. Permutationally concave games have a non-empty core, but the structure of that core is not known.

Kar (2002) provides a characterization of the Shapley value of C, giving his name to the solution. As C is not always monotonically increasing and concave, the Kar solution can allocate negative shares to some agents and can be outside of the core. While the non-stability of the Kar solution is a weakness, the possible negativity should not be considered so in the context of the game with property rights.

Bird (1976) introduced the concept of irreducible cost matrix, on which is built the most famous cost allocation for mcst problems. First suggested by Feltkamp et al. (1994) and rediscovered independently by Bergantinos and Vidal-Puga (2007), as well as being the average of the solutions proposed by Norde et al. (2004), we follow Bogomolnaia and Moulin (2010) and call it the folk solution. We obtain the irreducible cost matrix as follows: the cost of all edges are reduced as much as possible, with the constraint that the total cost of the project remains unchanged. Interestingly, for an irreducible matrix, there is no distinction between the stand-alone cost games with or without property rights. The folk solution is the Shapley value of the stand-alone cost game generated by this irreducible cost matrix. Since it does not matter if we consider the game with property rights or not, this has blurred the line between the two approaches. While Bogomolnaia and Moulin (2010) clearly choose the no property rights approach after discussing the two, other papers model the game with property rights but apply properties and axioms in the spirit of the no property rights approach, for instance non-negativity of the cost shares. For an irreducible matrix, C (and C^{NR}, as they are the same) is monotonically increasing and concave. As the cost is monotonically increasing, all incremental costs are also non-negative. As all incremental cost allocations are in the core and the Shapley value is the average of all incremental cost allocations, the Shapley value of a concave game is also in the core (Shapley (1967)). The folk solution is actually in $core(C^{NR})$ of the original cost matrix Bogomolnaia and Moulin (2010). Therefore, the folk solution, while applicable in the property rights context, is really more in line with the no rights approach.

For instance, Bergantinos and Vidal-Puga (2007, 2008)define a set of 12 properties on cost sharing solutions, with 7 of them satisfied by the folk solution but not the Kar solution. Most of these properties are based on ideas that are clearly compatible with the property rights approach, like Non-Negativity of the cost shares, Reductionism (two problems with the same optimal network configurations (including the cost of each edge in it) have the same cost allocations), Independence of Large Costs (allocation of agent i is independent of the cost of edges that are most expensive than all edges going through the node of agent i) and Solidarity (if the costs of all edges have weakly increased, all cost allocations should also weakly increased). The two main remaining properties are Core Selection and Population Monotonicity (adding an agent to the problem should not increase the cost share of any of the original agents).

The main goal of this paper is to show that it is possible to have a continuous[1] cost sharing solution that is a core selection and that is more in line with the property rights approach. In particular, no solution has been proposed that is both in $core(C)$ and takes into account the stand-alone costs of coalitions who can only use the locations of its members to connect to the source. More generally, we are looking for cost sharing solutions that are core selections and that do not throw away as much information as the folk solution, who relies on the irreducible matrix. While the fact that the Kar solution is not always a core selection is a sign that we cannot keep all of the information contained in the property rights game, it is not clear that we need to throw most of it as for the folk solution. In particular, if an agent i has a better location than an agent j, the folk solution might not allocate him strictly less than j. Similiarly, if the cost of the edge between i and j drops, i and j do not always pay strictly less than before. These properties of Cost Monotonicity and Ranking are satisfied by the Kar solution.

Bogomolnaia and Moulin (2010) perform a similar search of core selections satisfying Cost Monotonicity and Ranking, however restricted to non-negative solutions, as they use the no property rights approach. Their versions of these properties are restricted to cost matrices where the connection costs to the source are the most expensive. For most of their results, they also need to abandon Piecewise Linearity, a property close to the classic Additivity property.

In this paper, we offer, for the property rights approach, a solution that is both a core selection and satisfies stronger version of Cost Monotonicity and Ranking. First, we show a sufficient condition for C to be a concave game. This condition is closely related to the method used to compute the irreducible matrix, from which we get the folk solution. For any edge (i, j), we find the path between i and j for which its most expensive edge has the lowest value and assigns this value to edge (i, j) in the irreducible matrix. The condition for concavity is similar: we find the cycle that goes through i and j for which its most expensive edge has the lowest value. We compute a reduced matrix by assigning this value to edge (i, j) if it is smaller than the direct connection cost c_{ij}. Therefore, a cost matrix c for which the above method yields no transformation generates a concave game. It becomes a necessary condition for concavity if we consider an elementary cost matrix, where all edges have a cost of 0 or 1. A similar result was obtained by van den Nouweland and Borm (1991) in communication games. Another interpretation is that we reduce the cost of edge i, j only if there are two distinct coalitions that allow us to construct a path from i to j such that the most expensive edge in those paths is cheaper than a direct connection between i and j. Therefore, there are fewer reductions that occur compared to the irreducible cost matrix. In graph theory terms, we transform the cost matrix c such that for all $\alpha \in \mathbb{R}_+$, the graph consisting of edges that do not cost more than α is cycle-complete: the subgraph generated by agents contained in a cycle is complete. This transformation also has an interesting property: if the game is generated by an elementary cost matrix then the transformed game has the same core as the original game.

We then build a new cost sharing solution around this result by taking the Shapley value of this concave game. This generates a stable allocation. In addition, since there are fewer reductions, we obtain a rule that is more responsive to changes in cost than the folk solution. It is, of course, less responsive than the Kar solution, but it is now stable. In opposition to the Kar rule, it is easily computed.

However, we show that the property of Population Monotonicity is incompatible with all but the weakest of responsibility properties. We also show that stability and non-negativity do not necessarily go hand-in-hand, as the new solution proposed is not non-negative. We further examine the link between negativity, stability and responsibility properties.

[1]The Bird and the Dutta-Kar solutions (Dutta and Kar (2004)), for instance, are based on the Prim algorithm that finds the optimal network configurations. A small change in the cost of some edges can completely change the allocations.

The paper is as follows. In section 2, we define minimum cost spanning tree problems as well as the Kar and folk solutions. Section 3 gives the conditions for a mcst problem to generate a concave game. This result is used to define the cycle-complete cost matrix and a new cost sharing solution in section 4. That section also contains the links between the cores of the original game and of the game generated by the cycle-complete cost matrix. Section 5 discusses the links between stability properties and those making agents responsible for their location. Section 6 examines the link between Non-negativity and those properties.

2 Minimum cost spanning tree problems and common solutions

2.1 Mcst problems

Let $N = \{1, ..., n\}$ be the set of agents and let 0 denote the source to which agents have to be connected. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between i and j. Let $c = (c_{ij})_{i,j \in N_0^P}$ be a vector in $\mathbb{R}_+^{N_0^P}$ with c_{ij} representing the cost of edge (i, j). Let $\Gamma(N)$ be the set of all cost vectors for the set of agents N. Since c assigns a cost to all edges (i, j), we often abuse language and call c a cost matrix. A minimum cost spanning tree problem is a triple $(0, N, c)$. Since 0 does not change, we omit it in the following and simply identify a mcst problem as (N, c).

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree τ is identified by the set of its edges. Its associated cost is $\sum_{e \in \tau} c_e$.

Let p_{lm} be a path between l and m. It is a set of K distinct edges (i_k, i_{k+1}), with $k \in [0, K-1]$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $P_{lm}(N_0)$ be the set of all such paths between l and m when all locations in N_0 can be used. A cycle p_{ll} is a set of K distinct edges (i_k, i_{k+1}), with $k \in [0, K-1]$, $K \geq 3$ and such that $i_0 = i_K = l$.

We say that a path p_{lm} is free if $c_e = 0$ for all $e \in p_{lm}$. The same definition holds for free cycles. We say that an agent i is in the path (or cycle) p_{lm} if there exists $j \in N_0$ such that $(i, j) \in p_{lm}$. Finally, the notation p_{ij}^S indicates that the path from i to j uses only the locations of agents in $S \cup \{i, j\}$; i.e., that there is no agent in the path p_{ij}^S that is part of $N_0 \backslash \{S \cup \{i, j\}\}$.

Let $F_S(R)$ be the set of cycles in R that contain S, with $S \subseteq R \subseteq N_0$. We will often rank paths according to their most expensive arc. Let $\hat{p}_{ij}^R \in \arg\min_{p_{ij} \in P_{ij}(R)} \max_{e \in p_{ij}} c_e$ be (one of) the path between i and j that has the cheapest most expensive edge. Let $\hat{c}_{ij}^R = \min_{p_{ij} \in P_{ij}(R)} \max_{e \in p_{ij}} c_e$ be that cost. Similarly, let $\hat{f}_S^R \in \arg\min_{f \in F_S(R)} \max_{e \in f} c_e$ be (one of) the cycle(s) in R containing S with the smallest most expensive cost.

An edge (i, j) is said to be irrelevant if $c_{ij} > \max(c_{0i}, c_{0j})$ as it is never used, not even by coalition $\{i, j\}$. All common cost sharing solutions satisfy a property called Independence of Irrelevant Edges that says that no cost share depends on the cost of an irrelevant edge (Trudeau (2010)). Alternatively, we could consider only cost matrices that contain no such irrelevant edges (i.e. such that for all pairs (i, j), $c_{ij} \leq \max(c_{0i}, c_{0j})$).

The minimum cost of connecting N to the source and the associated minimum cost spanning tree is obtained using Prim's algorithm, which has n steps. First, pick an edge $(0, i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. In the second step, we choose an edge with the smallest cost connecting an agent in $N \backslash \{i\}$ either directly to the source or to i, which is connected. We continue until all agents are connected, at each step connecting an agent not already connected to the source or to an agent previously connected. Let $C(N, c)$ be the associated cost. Note that the mcst might not be unique.

Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$, and let $C(S, c)$ be the cost of the mcst of the problem (S, c^S). We say that C is the stand-alone cost function associated with c. Therefore, we say that C is the stand-alone cost game generated by the mcst problem (N, c) when we are in the property rights approach, as we don't allow agents in S to connect to the source through agents in $N\backslash S$. Let $C^{NR}(S,c)$ be the cost to connect all agents in S to the source, using all locations in N_0. C^{NR} is the stand-alone cost game generated by the mcst problem (N,c) when we are in the no property rights approach

2.2 Cost sharing solutions

A cost allocation $y \in \mathbb{R}^N$ assigns a cost share to each agent and the budget balance condition is $\sum_{i\in N} y_i = C(N,c)$. Note that these cost shares can be negative. Since C is not necessarily monotonic, we may be justified in subsidizing an agent. For any $S \subset N$, let $y^S = \sum_{i\in S} y_i$. A cost sharing solution (or rule) assigns a cost allocation $y(N,c)$ to any admissible mcst problem (N,c). When there is no risk of confusion, we use $y(c)$ instead of $y(N,c)$.

The Kar solution was explicitly defined and characterized in Kar (2002). It is the Shapley value of the game C. More precisely,

$$y_i^k(N,c) = Sh_i(C) = \sum_{S\subseteq N\backslash\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C(S\cup\{i\},c) - C(S,c)\right)$$

for all $i \in N$, with $C(\emptyset,c) = 0$.

As mentioned in the introduction, the so-called folk solution has been obtained in different ways. We focus on the approach of Bergantinos and Vidal-Puga (2007), which uses the Shapley value, thus allowing for a clear comparison with the Kar solution.

From any cost matrix c, we can define the irreducible cost matrix $\bar{c}$ as follows:

$$\bar{c}_{ij} = \min_{p_{ij}\in P_{ij}(N_0)} \max_{e\in p_{ij}} c_e = \hat{c}_{ij}^{N_0}.$$

Let $\bar{\Gamma}(N)$ be the set of all irreducible cost matrices.for the set of agents N. We have the following result.

Lemma 1 *For all $c \in \bar{\Gamma}(N)$ and all $S \subseteq N$, $C(S,c) = C^{NR}(S,c)$.*

Proof. Suppose that it is not true. Then, for some $S \subset N$, $i,j \in S_0$, there exists $p_{ij} \in P_{ij}(N_0)$ such that $\sum_{e\in p_{ij}} \bar{c}_e < \bar{c}_{ij}$. This is impossible as $\bar{c}_{ij} = \min_{p_{ij}\in P_{ij}(N_0)} \max_{e\in p_{ij}} c_e$ ■

The folk solution, y^f, is the Shapley value of the stand-alone cost function associated to $\bar{c}$, defined as $\bar{C}(S,c) = C(S,\bar{c}) = C^{NR}(S,\bar{c})$ for all $S \subseteq N$.

3 Concavity

Before defining the conditions for a msct problem to generate a non-concave cost game, we need the concepts of Piecewise Linearity and elementary matrices. Piecewise Linearity says that if we can decompose a cost matrix into submatrices where the cost of all edges are ordered in the same manner as the original matrix, then the cost allocation on the original cost matrix should equal the sum of the cost allocations on the submatrices. This property (or similar versions), a weaker version than the classic Additivity property in the general setting, has been used in Branzei et al. (2004), Tijs et al. (2005, 2006), Bergantinos and Vidal-Puga (2009) and Bogomolnaia and Moulin (2010). Piecewise Linearity generates a rich class of solutions having a simple structure. Cost shares can be defined on simple elementary matrices where costs of all edges are either 0 or 1, making it particularly appealing. In addition, many normative properties easily defined on those elementary matrices automatically extend to arbitrary matrices. Both the Kar and the folk solutions satisfy Piecewise Linearity. It is used in Trudeau (2010) to characterize the affine combination of these two solutions.

Denote arbitrarily the $p = \frac{n(n+1)}{2}$ distinct edges in N_0^P, such that $c = (c_{e^1},...,c_{e^p})$. For any permutation σ of $\{1,...,p\}$, define $K_\sigma = \{c \in \Gamma(N) \mid c_{e^{\sigma(1)}} \leq c_{e^{\sigma(2)}} \leq c_{e^{\sigma(p)}}\}$ to be the cone in $\Gamma(N)$

containing all cost matrices with a given increasing ordering of connection costs. Note that $\Gamma(N) = \cup K_\sigma$.

We said that a cost sharing rule satisfies Piecewise Linearity if it is linear in K_σ. More precisely, denote by Γ^E the set of elementary cost matrices where all connection costs are either 0 or 1 : $\Gamma^E(N) = \left\{c \in \Gamma(N) : c_e \in \{0,1\} \text{ for all } e \in N_0^P\right\}$. For any cone K_σ and any $k \in \{1,...,p\}$, let $b^k \in \Gamma^E(N)$ be such that $b^k_{e^{\sigma(1)}} = ... = b^k_{e^{\sigma(k-1)}} = 0$ while $b^k_{e^{\sigma(k)}} = ... = b^k_{e^{\sigma(p)}} = 1$. Then, a cost sharing rule is piecewise linear if

$$y(N,c) = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) y(N, b^k) \text{ for any } \sigma \text{ and any } c \in K_\sigma.$$

Recall that a game C is said to be concave if

$$C(S) + C(T) \geq C(S \cup T) - C(S \cap T)$$

for any $S, T \subseteq N$. Alternatively, we can rewrite this definition as

$$C(S \cup \{i\}) - C(S) \geq C(T \cup \{i\}) - C(T)$$

for any $S \subset T \subseteq N \backslash \{i\}$. We are now ready to state the condition under which an elementary cost matrix generates a non-concave cost game.

Theorem 1 *An elementary matrix c with no irrelevant links generates a non-concave stand-alone game C if and only if there exists a free cycle f and $i, j \in N_0$ in this cycle for which $c_{ij} = 1$.*

Proof. Let's first show that if there exists a free cycle f and a pair of agents i, j in this cycle for which $c_{ij} = 1$, then the resulting stand-alone game is non-concave.

First, observe that this free cycle going through i and j implies that i) $c_{ij} = 1$, ii) $max\,(c_{0i}, c_{0j}) = 1$ and iii) there exists S, T disjoint in $N \backslash \{i,j\}$ such that we have free paths $p_{ij}^{S_0}$ and $p_{ij}^{T_0}$. Without loss of generality, suppose that S is such that for all $k \in S$, there exists $l \in S_0 \cup \{i,j\}$ such that $(k,l) \in p_{ij}^{S_0}$ (i.e., that S is composed of all agents that are in the path $p_{ij}^{S_0}$). Make the same assumption for T.

As c is fixed, to simplify the notation, we write $C(S)$ instead of $C(S,c)$.

Now, we examine the case where $i = 0$ and $j \in N$.

Then, clearly, $C(\{j\}) = 1$. Consider now $C(R \cup \{j\})$, with $R \in \{S, T\}$. By definition, there is a free path $p_{0j}^{R_0}$ that includes all agents in R. Therefore, $C(R \cup \{j\}) = 0$. Clearly, we also have that $C(S \cup T \cup \{j\}) = 0$.

Therefore,

$$-1 = C(S \cup \{j\}) - C(\{j\}) < C(S \cup T \cup \{j\}) - C(T \cup \{j\}) = 0$$

Next, we consider the case where $i, j \in N$. Let's consider first $C(\{i,j)\}$. Using Prim's algorithm, we first pick the cheapest arc between $(0,i)$ and $(0,j)$. Then, we connect the remaining agent. By i) and ii), $c_{ij} = 1$ and $\max\,(c_{0i}, c_{0j}) = 1$. Therefore, connecting that second agent always costs 1 and $C(\{i,j)\} = \min\,(c_{0i}, c_{0j}) + 1$.

Consider now $C(R \cup \{i,j\})$, with $R \in \{S,T\}$. We first pick the cheapest arc connecting an agent in $R \cup \{i,j\}$ to the source. Then, by iii) and our assumption that all agents in R are in the path p_{ij}^{R}, we have arcs with zero cost connecting all the remaining agents. Therefore, $C(R \cup \{i,j\}) = \min_{k \in R \cup \{i,j\}} c_{0k}$.

Similarly, $C(S \cup T \cup \{i,j\}) = \min_{k \in S \cup T \cup \{i,j\}} c_{0k}$ as we have paths connecting agents in S and agents in T at zero cost.

We then show that

$$C(S \cup \{i,j\}) - C(\{i,j\}) < C(S \cup T \cup \{i,j\}) - C(T \cup \{i,j\})$$

or

$$\min_{k \in S \cup \{i,j\}} c_{0k} - \min_{k \in \{i,j\}} c_{0k} - 1 < \min_{k \in S \cup T \cup \{i,j\}} c_{0k} - \min_{k \in T \cup \{i,j\}} c_{0k}.$$

Case a) If $\min_{k\in\{i,j\}} c_{0k} = 0$, then $\min_{k\in S\cup\{i,j\}} c_{0k} = \min_{k\in T\cup\{i,j\}} c_{0k} = \min_{k\in S\cup T\cup\{i,j\}} c_{0k} = 0$, and we have that $-1 < 0$.

Case b) If $\min_{k\in\{i,j\}} c_{0k} = 1$ and $\min_{k\in S\cup\{i,j\}} c_{0k} = 0$, then $\min_{k\in S\cup T\cup\{i,j\}} c_{0k} = 0$, and $-2 < -\min_{k\in T\cup\{i,j\}} c_{0k}$, since $\min_{k\in T\cup\{i,j\}} c_{0k}$ cannot be higher than 1.

Case c) If $\min_{k\in\{i,j\}} c_{0k} = \min_{k\in S\cup\{i,j\}} c_{0k} = 1$, then $\min_{k\in T\cup\{i,j\}} c_{0k} = \min_{k\in S\cup T\cup\{i,j\}} c_{0k}$ and we have $-1 < 0$.

Therefore, in all cases, we have $C(S\cup\{i,j\}) - C(\{i,j\}) < C(S\cup T\cup\{i,j\}) - C(T\cup\{i,j\})$, which proves non-convavity.

Next, we show that if an elementary matrix c generates a non-concave stand-alone game (N, C) then there exists a free cycle f and $i, j \in N_0$ in this cycle for which $c_{ij} = 1$.

If C is non-concave, then there exists $S \subset T \subset N$ and $k \in N\backslash T$ such that $C(S\cup\{k\}) - C(S) < C(T\cup\{k\}) - C(T)$.

Suppose that $C(S\cup\{k\}) - C(S) = 1$. This implies that $c_{ik} = 1$ for all $i \in S\cup\{0\}$. Then, $C(T\cup\{k\}) - C(T) \le 1$, which contradicts our non-concavity assumption.

Suppose that $C(S\cup\{k\}) - C(S) = 0$. This implies that there exists $i \in S$ such that $c_{ik} = 0$. Then, $C(T\cup\{k\}) - C(T) \le 0$, as $i \in S \subset T$, which contradicts our non-concavity assumption.

Therefore, $C(S\cup\{k\}) - C(S) < 0$. This implies that agent k allows at least one pair of agents (or an agent and the source) to connect to each other through a free path, something that they could not do without k. Formally, this implies that there exists $i, j \in S_0$ and $p_{ij}^{S_0\cup\{k\}}$ such that $c_e = 0$ for all $e \in p_{ij}^{S_0\cup\{k\}}$ and that there are no paths $p_{ij}^{S_0}$ such that $c_e = 0$ for all $e \in p_{ij}^{S_0}$.

Since $C(S\cup\{k\}) - C(S) < C(T\cup\{k\}) - C(T)$, we have $i, j \in S_0$ such that there exists free paths $p_{ij}^{S_0\cup\{k\}}$, $p_{ij}^{T_0}$ and $p_{ij}^{T_0\cup\{k\}}$, but no free path $p_{ij}^{S_0}$. Now, construct the graph $Z = p_{ij}^{S_0\cup\{k\}} \cap p_{ij}^{T_0}$. We show that Z contains at least one free cycle. The only way that Z would not contain any free cycle is if $p_{ij}^{S_0\cup\{k\}} \subseteq p_{ij}^{T_0}$ or $p_{ij}^{T_0} \subseteq p_{ij}^{S_0\cup\{k\}}$. However, this is impossible, because there is no free path $p_{ij}^{S_0}$, so $p_{ij}^{S_0\cup\{k\}}$ must go through agent k and $k \notin T_0$. Label Z^{fc} that free cycle in Z.

The path $p_{ij}^{S_0\cup\{k\}}$ connects i and j through agent k and n_{ij}^S agents in $S\backslash\{i,j\}$, with $n_{ij}^S \in [0, |S|-2]$. The path $p_{ij}^{T_0}$ connects i and j through n_{ij}^T agents in $T\backslash\{i,j\}$, with $n_{ij}^T \in [1, |T|-2]$. This path cannot be made entirely of agents in S_0, since there is no free path $p_{ij}^{S_0}$.

Suppose that $n_{ij}^S = 0$, that is $p_{ij}^{S_0\cup\{k\}} = \{(i,k),(k,j)\}$. Then, Z^{fc} must go through i and j. Since $c_{ij} = 1$, we have shown that there exists a free cycle Z^{fc} of which i, j are a part of and for which $c_{ij} = 1$.

If $n_{ij}^S > 0$, then Z^{fc} goes through agents $l, m \in S$ (but not necessarily i and j), agent k and at least one agent $p \in T\backslash S$. We must show that at least one link between them is not free.

If $l, m \in \{i,j\}$, we are done, as $c_{ij} = 1$.

If $l = i$ and $m \in S_0\backslash\{i,j\}$, there exists a common free path between m and j that uses only agents in S. Then, we must have that $c_{im} = 1$. Otherwise, we would have a free path $p_{ij}^{S_0}$.

If $l, m \in S_0\backslash\{i,j\}$, there exists a common free path between l and i and one between m and j, with both using only agents in S. We must have $c_{lm} = 1$. Otherwise, we would have a free path $p_{ij}^{S_0}$. ■

The result can be restated as follows: an elementary cost matrix generates a non-concave game if and only if there exists a pair of agents (i,j) for which the direct connection cost is $c_{ij} = 1$ and there are two distinct subsets S and T in $N_0\backslash\{i,j\}$ that they can use to connect to each other freely. This is what generates the non-concavity.

We now extend this result to more general cost matrices.

Theorem 2 *A cost matrix c that generates a non-concave stand-alone game (N, C) contains at least one cycle f and a pair of agents i, j in this cycle for which $c_{ij} > \max_{e\in f} c_e$.*

Proof. We can show that $C(S, c) = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) C(S, b^k)$; i.e., that the stand-alone cost is piecewise-linear. If c contains at least one cycle f and a pair of agents i, j in this cycle for which $c_{ij} > \max_{e\in f} c_e$, then there exists at least one b^k to which a strictly positive weight $c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}$ is applied that appears in the piecewise-linear decomposition of c and that contains a free cycle f, with i,j in this cycle for which $b_{ij}^k = 1$.

We prove this result by contradiction. Suppose that, for any elementary matrix b^k to which a strictly positive weight is applied in the piecewise-linear decomposition of c, we have, for any free cycle f, no agents i,j in this cycle for which $b_{ij}^k = 1$. By Theorem 1, $C(N, b^k)$ is a concave game, and $C(S \cup \{i\}, b^k) - C(S, b^k) \geq C(T \cup \{i\}, b^k) - C(T, b^k)$ for all $S \subset T \subset N$ and $i \in N \backslash T$. Since $C(S, c) = \sum_{k=1}^{p} (c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}) C(S, b^k)$, we have $C(S \cup \{i\}, c) - C(S, c) \geq C(T \cup \{i\}, c) - C(T, c)$ for all $S \subset T \subset N$ and $i \in N \backslash T$. Therefore, the game C is concave, a contradiction with our assumption. ■

Note that this is a necessary but not a sufficient condition for non-concavity. Or, we can say that having no cycle f that contains a pair of agents i,j for which $c_{ij} > \max_{e \in f} c_e$ is a sufficient but not necessary condition for concavity.

3.1 Graph theory interpretation

We reinterpret the results above in terms of graph theory. First, we define a few notions.

A graph is a set of edges $G \subseteq (N_0)^P$. Suppose that G contains a cycle $((i_0, i_1), (i_1, i_2), ..., (i_K, i_0))$. Let $S = \{i_0, i_1, ..., i_K\}$. We say that G is *cycle-complete* if for all $i, j \in S$, the edge $(i,j) \in G$.[2]

Given a cost matrix c, we define $G(c, \alpha)$ as follows: for any $e \in (N_0)^P$, if $c_e \leq \alpha$, then $e \in G(c, \alpha)$, for $\alpha \in \mathbb{R}_+$.

We can then restate Theorem 1 as follows: an elementary matrix c with no irrelevant links generates a concave stand-alone game C if and only if $G(c, 0)$ is cycle-complete.

Theorem 2 can now be expressed in this manner: If a cost matrix c generates a non-concave stand-alone game (N, C), then there exists $\alpha \in \mathbb{R}_+$ such that $G(c, \alpha)$ is not cycle-complete. If $G(c, \alpha)$ is cycle-complete for all $\alpha \in \mathbb{R}_+$, then c generates a concave stand-alone game.

A similar result was obtained by van den Nouweland and Borm (1991) in communication games, where agents gain from being able to communicate with others. A graph summarizes the various communication possibilities. They found a strong link between the cycle-completeness of the communication graph and the convexity of the resulting game.

4 The cycle-complete cost matrix, the associated core and a new cost sharing solution

The folk solution is built around the concept of reduction of the cost matrix. To obtain the irreducible matrix, for an edge (i,j), we find the path between i,j for which its most expensive edge is as cheap as possible and assign this cost to (i,j). The resulting game is concave and monotonically increasing.

From the previous section, we now have a reason to do a different kind of reduction.

4.1 The cycle-complete cost matrix

We find the cycle that includes i and j and for which its most expensive edge is as cheap as possible and assign this cost to (i,j). Alternatively, we can say that we modify the game to make it such that $G(c, \alpha)$ is cycle-complete for all $\alpha \in \mathbb{R}_+$. Let c^* be the *cycle-complete cost matrix*: for all $i, j \in N_0$, $c_{ij}^* = \min\left(c_{ij}, \max_{e \in \hat{f}_{\{i,j\}}^{N_0}} c_e\right)$. The resulting game $C^*(N, c) = C(N, c^*)$ is concave, but not necessarily monotonically increasing.

For each pair (i,j), the cycle-complete cost matrix ranks cycles going through i and j according to their cheapest edge and assigns to the pair the smallest of those values. By comparison, the irreducible matrix does the same thing, but with paths instead of cycles.

[2] Stated otherwise, all members of a cycle must form a clique.

In conceptual terms, we can see the paths as technologies to connect i to j. In the irreducible matrix, we consider that i and j can always use the best technology. The cycle-complete cost matrix lets i and j use that best available technology only if it can obtain it from two distinct sources: it then enters the public knowledge. The Kar solution, which uses the original cost matrix, supposes that this technology can only be used if there is a cooperation with the owners of that technology (the agents through which that path goes through).

4.2 Core of C^*

It turns out that the core of C^* is closely related to the core of C. In fact, for elementary matrices, both games have the same core. To prove this, we need the following definition.

A connected component in the graph G is a set of agents S (or a set of agents and the source) such that for all $i, j \in S$, there exists a free path p_{ij}^S in the graph and such that all $T \supset S$ are not a connected component. Also, if i is part of no free path, then $\{i\}$ is a connected component. It is well known that for an elementary matrix, $C(S,c)$ is equal to the number of connected components in $G(c^S, 0)$ minus one.

Theorem 3 *If C is a cost game generated by a mcst problem with an elementary cost matrix with no irrelevant edge, $core(C) = core(C^*)$.*

Proof. By definition, $C^*(S,c) \leq C(S,c)$ for all S. Therefore, $core(C^*) \subseteq core(C)$. Since C^* is concave, $core(C^*)$ is exact (Schmeidler (1972)).

Suppose $core(C) \subsetneq core(C^*)$, that $C(S,c) > C^*(S,c)$ and that there exists $y \in core(C)$ such that $y(S) > C^*(S,c)$.

Suppose that S can be partitioned in $(S_1, ..., S_K)$ such that if $i \in S_k$ and $j \in S_l$, then i and j are in different connected components of $G(c,0)$. It is easy to see that $C(S,c) = \sum_{k=1}^{K} C(S_k, c)$. Therefore, suppose that T (or T_0) is a connected component of $G(c,0)$. The constraints for x to be in $core(C(\cdot,c))$ relevant to agents in T are the same as those for $core(C(\cdot, c^T))$. Therefore, we can consider a coalition S such that all agents in S belong to the same connected component T of $G(c,0)$.

The fact that $C(S,c) > C^*(S,c)$ implies that there exists $i, j \in S_0$ such that $c^*_{ij} = 0$ and $c_{ij} = 1$. Stated otherwise, there exist R^1, R^2 distinct in S such that we have free paths $p_{ij}^{R_0^1}$ and $p_{ij}^{R_0^2}$.

For simplification, suppose that S is partitioned in two. If $0 \in T$, then the connected components of $G(c^S, 0)$ are S_1 and $(S_2)_0$ with $i \in S_1$ and $j \in (S_2)_0$. If $0 \notin T$, then the connected components of $G(c^S, 0)$ are $\{0\}$, S_1 and S_2 with $i \in S_1$ and $j \in S_2$. The following result can be extended to more general partitions.

We define T_1 and T_2 in the following manner:

For all $k \in T \backslash S$,

$$\begin{aligned} k &\in T_1 \text{ if } k \in R_1 \text{ or if } k \notin R_2 \text{ and there exists } l \in R_1 \text{ such that } c_{kl} = 0 \\ k &\in T_2 \text{ otherwise} \end{aligned}$$

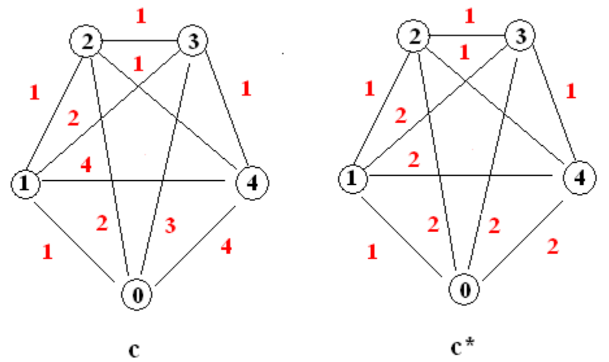
By definition, $S \cup T_1 \cup T_2 = T$. We also have that $C(S \cup T_1, c) = C(S \cup T_2, c) = C(T,c) = C(S,c) - 1 = C^*(S,c)$.

We have $y(S) = C(S,c) - 1 + \epsilon > C^*(S,c)$, with $\epsilon > 0$. The core restrictions imply that

$$\begin{aligned} y(S \cup T_1) &\leq C(S,c) - 1 \\ y(S \cup T_2) &\leq C(S,c) - 1 \\ y(T) &= C(S,c) - 1 \end{aligned}$$

Therefore, we need $y(T_1) \leq -\epsilon$, $y(T_2) \leq -\epsilon$ and $y(T_1 \cup T_2) = -\epsilon$. Since $T_1 \cap T_2 = \emptyset$, the only possibility is $\epsilon = 0$. We cannot have $y(S) > C^*(S,c)$. Therefore, $core(C) \subseteq core(C^*)$. ∎

Figure 1: Example showing that $core(C) \subsetneq core(C^*)$



S	$C(S,c)$	$C^*(S,c)$	S	$C(S,c)$	$C^*(S,c)$
$\{1\}$	1	1	$\{2,3\}$	3	3
$\{2\}$	2	2	$\{2,4\}$	3	3
$\{3\}$	3	2	$\{3,4\}$	4	3
$\{4\}$	4	2	$\{1,2,3\}$	3	3
$\{1,2\}$	2	2	$\{1,2,4\}$	3	3
$\{1,3\}$	3	3	$\{1,3,4\}$	4	4
$\{1,4\}$	5	3	$\{2,3,4\}$	4	4
			$\{1,2,3,4\}$	4	4

This result allows for a precise way to find the core of the game generated by a mcst problem if the cost matrix is elementary. Since C^* is concave, the incremental cost shares constitute the extreme points of $core(C^*)$ and thus of $core(C)$.

However, this result is not true for more general cost matrices.

Theorem 4 *In general, if C is a cost game generated by a mcst problem, $core(C^*) \subseteq core(C)$ but $core(C) \subsetneq core(C^*)$.*

The first part is clear as $C^*(S,c) \leq C(S,c)$ for all S and c. We can see the second part with the example in Figure 1, where the cost allocation $(0,0,1,3)$ is in $core(C)$ but not in $core(C^*)$.

4.3 A new cost sharing solution

Having defined a concave game that is closely related to the original game, the next step is to build a cost sharing solution from it, that we label the cycle-complete cost sharing solution, or y^{CC}. We define $y^{CC}(N,c) = Sh(C(N,c^*))$. However, the set of cycles that include i and j is large and not easy to work with. We offer an alternative definition based on irreducible matrices, that are relatively easy to compute. Let $\bar{c}^{N\setminus\{i\}}$ be the irreducible matrix associated to $c^{N\setminus\{i\}}$. Define $\mathring{c}$ such that for $i,j \in N$, $\mathring{c}_{ij} = \max_{k\in N\setminus\{i,j\}} \bar{c}_{ij}^{N\setminus\{k\}}$ and for $i \in N$, $\mathring{c}_{0i} = \max_{k\in N\setminus\{i\}} \bar{c}_{ij}^{N\setminus\{k\}}$. We show that $\mathring{c}$ are c^* are the same and thus that $y^{CC}(N,c) = Sh(C(N,\mathring{c}))$.

Lemma 2 *For all N and all $c \in \Gamma(N)$, we have $\mathring{c} = c^*$.*

Proof. We first show that if (i,j) is such that $c_{ij} > \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$, then $\mathring{c}_{ij} = \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e = c^*_{ij}$. We then show that in all other cases $\mathring{c}_{ij} = c_{ij} = c^*_{ij}$.

Suppose that $c_{ij} > \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$. Take $S \in \arg\min_{S \subset N_0 \setminus \{i,j\}} \max \left\{ \hat{c}_{ij}^{S_0 \cup \{ij\}}, \hat{c}_{ij}^{(N \setminus S)_0} \right\}$. Then, there exist two distinct subsets S_0 and $(N \setminus S)_0$ of $N_0 \setminus \{i,j\}$ and paths $\hat{p}_{ij}^{S_0}$ and $\hat{p}_{ij}^{(N \setminus S)_0}$ such that $\max_{e \in \hat{p}_{ij}^{R_0}} < c_{ij}$ for $R \in \{S, N \setminus S\}$. Then, $\max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e = \max \left\{ \hat{c}_{ij}^{S_0 \cup \{ij\}}, \hat{c}_{ij}^{(N \setminus S)_0} \right\}$. Suppose that $\hat{c}_{ij}^{S_0 \cup \{ij\}} \geq \hat{c}_{ij}^{(N \setminus S)_0}$.

Take $k \in S$ and consider the cost matrix $c^{N \setminus \{k\}}$. Clearly, we can still connect agent i to j through $\hat{p}_{ij}^{(N \setminus S)_0}$ and therefore $\bar{c}_{ij}^{N \setminus \{k\}} \leq \hat{c}_{ij}^{(N \setminus S)_0}$.

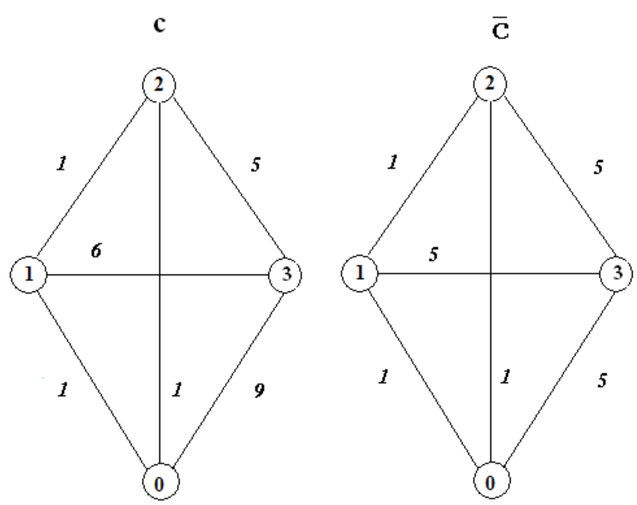
Take $k \in N \setminus S$ and consider the cost matrix $c^{N \setminus \{k\}}$. Clearly, we can still connect agent i to j through $\hat{p}_{ij}^{S_0}$ (but we possibly can do better by connecting them through R, with $R \subseteq N_0 \setminus \{k\}$) and therefore $\bar{c}_{ij}^{N \setminus \{k\}} \leq \hat{c}_{ij}^{S_0 \cup \{ij\}}$. Suppose that for all $k \in N \setminus S$, $\bar{c}_{ij}^{N \setminus \{k\}} < \hat{c}_{ij}^{S_0 \cup \{ij\}}$. This implies that there is a cycle going through i, j for which the highest cost is smaller than $\hat{c}_{ij}^{S_0 \cup \{ij\}}$, thus contradicting our assumption. Therefore, if $c_{ij} > \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$, then $\mathring{c}_{ij} = \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$. By definition, this is equal to c^*_{ij}.

Suppose that $c_{ij} \leq \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$. Then, $c^*_{ij} = c_{ij}$. Let $\bar{P}_{ij}$ represent the set of paths from i to j such that if $p_{ij} \in \bar{P}_{ij}$, then $\max_{e \in p_{ij}} c_e < c_{ij}$. Let $A(p_{ij})$ be the set of agents (other than i and j) that are in p_{ij}; i.e., if $k \in A(p_{ij})$ there exists l such that $(k,l) \in p_{ij}$ If $\bar{P}_{ij} = \emptyset$, then clearly $\mathring{c}_{ij} = c_{ij}$. If $\bar{P}_{ij} \neq \emptyset$, by the assumption that $c_{ij} \leq \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$, we have that $\cap_{p_{ij} \in \bar{P}_{ij}} A(p_{ij}) \neq \emptyset$. Therefore, the presence of (at least) one agent, say k, is needed for i and j to improve on their direct connection. Therefore, $\bar{c}_{ij}^{N \setminus \{k\}} = c_{ij}$, which implies that $\mathring{c}_{ij} = c_{ij} = c^*_{ij}$. ■

This gives us alternative definition for the cycle-complete solution, based on the concept of irreducible matrix. The following example help show the differences between y^{CC} and the Kar and folk solutions.

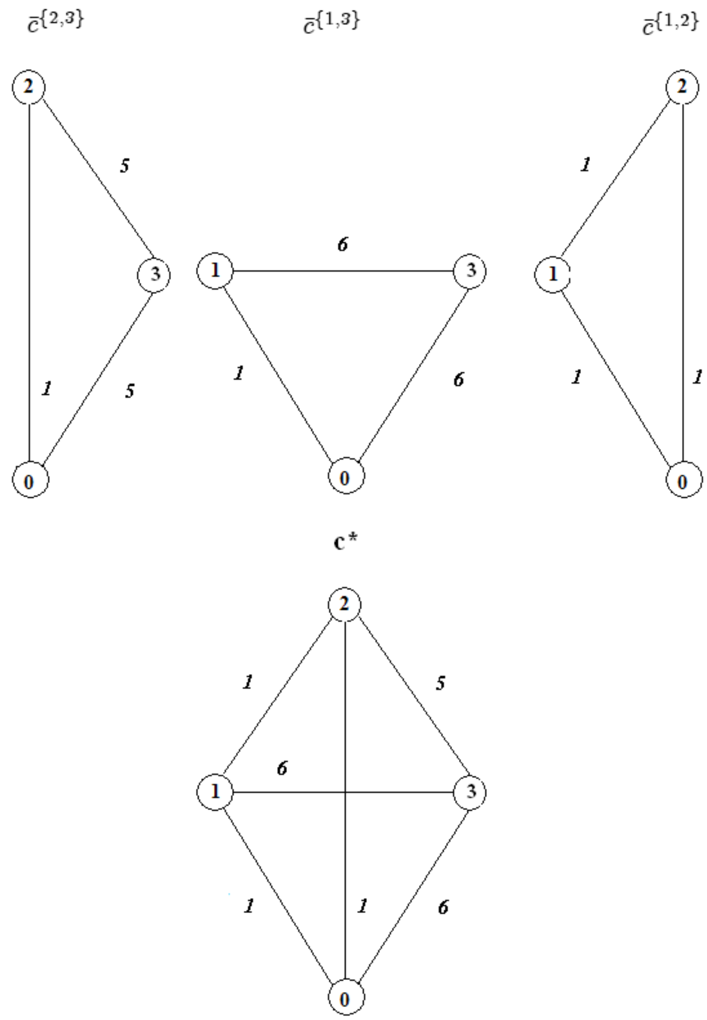
Example 1 *Consider the following three-player example and the cost matrix c. Agents are identified in the circles, with costs next to edges. On the right, we find the associated irreducible matrix $\bar{c}$.*

Figure 2: Cost matrix c and its irreducible matrix $\bar{c}$



We can see that there is a cycle $f = \{(0,1),(1,3),(3,2),(2,0)\}$ for which the most expensive edge has a cost of 6. Therefore, we will have that $c^*_{03} = 6$. Another way to compute c^* is to use the method *above with irreducible matrices. Removing one agent at a time, we obtain the irreducible matrices found in Figure 2. Taking the maximum value for each edge, we obtain c^*.*

Figure 3: Computation of c^*



The corresponding cost games are as follows.

S	$C(S,c)$	$C(S,\bar{c})$	$C(S,c^*)$
$\{1\}$	1	1	1
$\{2\}$	1	1	1
$\{3\}$	9	5	6
$\{1,2\}$	2	2	2
$\{1,3\}$	7	6	7
$\{2,3\}$	6	6	6
$\{1,2,3\}$	7	7	7

From this, we apply the Shapley value and obtain $y^k(N,c) = \left(\frac{1}{2}, 0, 6\frac{1}{2}\right)$, $y^f(N,c) = (1,1,5)$ *and* $y^{CC}(N,c) = \left(1, \frac{1}{2}, 5\frac{1}{2}\right)$.

Observe that the Kar solution is not in the core in this example, whereas the other two solutions are. We have that $y_2^k + y_3^k = 6\frac{1}{2} > 6 = C(\{2,3\}, c)$. Notice also that for all e, $\bar{c}_e \leq c_e^* \leq c_e$, something that holds in general. It also implies that $C(S,\bar{c}) \leq C(S,c^*) \leq C(S,c)$ for all S.

We next show formally that y^{CC} is a piecewise-linear solution that always yields an allocation in the core.

Core Selection: for all problems (N,c) and all $S \subseteq N$, $\sum_{i\in S} y_i(N,c) \leq C(S,c)$.

Lemma 3 *y^{CC} satisfies Piecewise Linearity and Core Selection.*

Proof. Piecewise Linearity: it suffices to show that $c_e^* = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) \left(b_e^k\right)^*$. If $c_{ij} \leq \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$, then because the ranking of the edges stay the same, we have $c_{ij}^* = c_{ij}$ and $\left(b_{ij}^k\right)^* = b_{ij}^k$. Thus, $c_{ij}^* = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) \left(b_{ij}^k\right)^*$

Suppose that $c_{ij} > \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$ and that the edge that has a cost of $\max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$ is ranked l in σ, while the edge (i,j) is ranked m. For all $k \leq l$, $b_{ij}^k = \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} b_e^k = 1$, and thus $\left(b_{ij}^k\right)^* = 1$. For $m > k > l$, we have $\max_{e\in \hat{f}^{N_0}_{\{i,j\}}} b_e^k = 0$ and $b_{ij}^k = 1$. Therefore, $\left(b_{ij}^k\right)^* = 0$. Combining these results, we have that $c_{ij}^* = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) \left(b_{ij}^k\right)^* = \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$.

Core Selection: By Lemma 2, $c_{ij}^* \leq \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$ for all pairs i,j. Therefore, by Theorem 2, C^* is concave. Since $y^{CC}(N,c) = Sh(C^*(N,c))$, the Shapley value of the concave game C^* is in the core of C^* (Shapley (1967)). Since $C(\cdot,c^*) \leq C(\cdot,c)$, y^{CC} satisfies Core Selection. ■

In addition, while computing C and its Shapley value is computationally very demanding, computing C^* and the corresponding Shapley value can be done in polynomial time. We say that G is a chordal graph if, for all cycles of four or more nodes, there exists a chord, that is an edge connecting two non-adjacent nodes in the cycle. Ando (2010) shows that computing C and its Shapley value can be done in polynomial time if $G(c,\alpha)$ is a chordal graph for all $\alpha \in \mathbb{R}_+$. Since c^* is such that $G(c^*,\alpha)$ is cycle-complete, it is clearly also a chordal graph.

5 Responsibility for one's location and stability

In this section, we consider properties that reward agents that are well located in the network. We examine which ones are respected by the folk and Kar solutions as well as our new solution defined in the previous section. We also look at compatibility issues between these properties and stability properties.

First, we examine different versions of the Cost Monotonicity property that conveys the simple idea that if the cost of an arc (i,j) decreases, agents i and j should also see their cost allocations decrease.

Cost Monotonicity$_1$: Suppose that $c_{ij} \leq \max\left[c_{0i}, c_{0j}\right]$. If c, c' such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(N,c') < y_k(N,c)$ for $k \in \{i,j\}$.

Cost Monotonicity$_2$: Suppose that $c_{ij} \leq \max_{e\in \hat{f}^{N_0}_{\{i,j\}}} c_e$. If c, c' such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(N,c') < y_k(N,c)$ for $k \in \{i,j\}$.

Cost Monotonicity$_3$: Suppose that $c_{ij} \leq \hat{c}_{ij}^{N_0}$. If c, c' such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(N,c') < y_k(N,c)$ for $k \in \{i,j\}$.

The first version always applies, except in the case where (i,j) is an irrelevant arc. The second version applies only when (i,j) has the same cost in the cycle-complete cost matrix and in the original one. The third version applies only when (i,j) has the same cost in the irreducible cost matrix and in the original one.

Lemma 4 *If a cost sharing solution satisfies Cost Monotonicity$_1$, it satisfies Cost Monotonicity$_2$. If a cost sharing solution satisfies Cost Monotonicity$_2$, it satisfies Cost Monotonicity$_3$.*

The proof is trivial as $\max[c_{0i}, c_{0j}] \geq \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e \geq \hat{c}^{N_0}_{ij}$.

Secondly, we study different versions of the Strict Ranking property, which says that if i has a strictly better location than j, then it should pay strictly less than j.

Strict Ranking$_1$: Suppose that $c_{ik} \leq c_{jk}$ for all $j \in N_0 \backslash \{i,j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0 \backslash \{i,j\}$, with $c_{il} < \max[c_{0i}, c_{0l}]$. Then $y_i(N,c) < y_j(N,c)$.

Strict Ranking$_2$: Suppose that $c_{ik} \leq c_{jk}$ for all $j \in N_0 \backslash \{i,j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0 \backslash \{i,j\}$, with $c_{il} < \max_{e \in \hat{f}^{N_0}_{\{i,l\}}} c_e$. Then $y_i(N,c) < y_j(N,c)$.

Strict Ranking$_3$: Suppose that $c_{ik} \leq c_{jk}$ for all $j \in N_0 \backslash \{i,j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0 \backslash \{i,j\}$, with $c_{il} < \hat{c}^{N_0}_{il}$. Then $y_i(N,c) < y_j(N,c)$.

The restrictions applied to the different versions are the same as those defined above for Cost Monotonicity.

Lemma 5 *If a cost sharing solution satisfies Strict Ranking*$_1$*, it satisfies Strict Ranking*$_2$*. If a cost sharing solution satisfies Strict Ranking*$_2$*, it satisfies Strict Ranking*$_3$*.*

Again, the proof of this lemma is trivial.

We consider another stability property, Population Monotonicity, that says that no agent should suffer from the addition of an agent to the project. This is important as it makes sure that no agent would veto such an addition.

Population Monotonicity: Let $c \in \Gamma(N)$ and $i \in S \subset N$. Then, $y_i(N,c) \leq y_i(S, c^S)$.

We then have the following lemma, that states the various properties satisfied by the cycle-complete solution, the folk solution and the Kar solution.

Lemma 6 *i) The folk solution satisfies Core Selection, Cost Monotonicity*$_3$*, Strict Ranking*$_3$ *and Population Monotonicity, but it fails to satisfy stronger versions of Cost Monotonicity and Strict Ranking.*

ii) The cycle-complete solution satisfies Core Selection, Cost Monotonicity$_2$ *and Strict Ranking*$_2$*, but it fails to satisfy Population Monotonicity, Cost Monotonicity*$_1$ *and Strict Ranking*$_1$*.*

iii) The Kar solution satisfies Cost Monotonicity$_1$ *and Strict Ranking*$_1$*, but it fails to satisfy Core Selection and Population Monotonicity.*

Proof. If a solution is $Sh(\theta)$, with θ a cost game, by the properties of the Shapley value, y_i is strictly increasing in $\theta(S)$ if $i \in S$. The Kar solution is $Sh(C)$, and if (i,j) is not irrelevant, a reduction in the cost c_{ij} will reduce some $C(S,c)$, with $\{i,j\} \subseteq S$ (most certainly it affects $C(\{i,j\},c)$). Therefore, the Kar solution satisfies Cost Monotonicity and Strict Ranking. In the same manner, a reduction in the cost of (i,j) will reduce $C(\{i,j\},c^*)$ if $c_{ij} \leq \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$. Therefore, y^{CC} satisfies Cost Monotonicity$_2$ and Strict Ranking$_2$. Also in the same manner, a reduction in the cost of (i,j) will reduce $C(\{i,j\},\bar{c})$ if $c_{ij} < \hat{c}^{N_0}_{ij}$. Therefore, the folk solution satisfies Cost Monotonicity$_3$ and Strict Ranking$_3$.

Bergantinos and Vidal-Puga (2007) show that the folk solution satisfies Core Selection and Population Monotonicity. We can see that for all $S \ni i$, $C(S,\bar{c})$ is not responsive to a small change of c_{ij} if $\max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e > c_{ij} > \hat{c}^{N_0}_{ij}$. Therefore, it does not satisfy Cost Monotonicity$_2$ and Strict Ranking$_2$.

Lemma 3 showed that y^{CC} satisfies Core Selection. We can see that for all $S \ni i$, $C(S,c^*)$ is not responsive to a small change of c_{ij} if $\max[c_{0i}, c_{0j}] > c_{ij} > \max_{e \in \hat{f}^{N_0}_{\{i,j\}}} c_e$. Therefore, it does not satisfy Cost Monotonicity$_1$ or Strict Ranking$_1$. Consider the problems $(\{1,2\},c)$ and $(\{1,2,3\},c')$ such that $c_{01} = c_{12} = 0$, $c_{02} = 1$ and $c'_e = c_e$ for $e \in \{0,1,2\}^p$, $c'_{03} = c'_{13} = c'_{23} = 0$. Solving, $y_1^{CC}(\{1,2\},c) = -\frac{1}{2}$ and $y_1^{CC}(\{1,2,3\},c') = 0$. Therefore, y^{CC} does not satisfy Population Monotonicity.

Bergantinos and Vidal-Puga (2007) show that the Kar solution does not satisfy Core Selection. From the previous example, $y_1^k(\{1,2\},c) = -\frac{1}{2}$ and $y_1^k(\{1,2,3\},c') = -\frac{1}{6}$. Therefore, the Kar solution does not satisfy Population Monotonicity. ■

All solutions satisfy Symmetry, which is defined as follows:

Symmetry: For any $c \in \Gamma(N)$, if i,j are such that $c_{ik} = c_{jk}$ for all $k \in N_0 \backslash \{i,j\}$, then $y_i(N,c) = y_j(N,c)$.

The Kar solution satisfies the strongest versions of Cost Monotonicity and Strict Ranking, but this comes at the price of Core Selection and Population Monotonicity. The folk solution satisfies Core Selection and Population Monotonicity, but this comes at the cost of having to weaken considerably Cost Monotonicity and Strict Ranking. The cycle-complete solution lies in between these two alternatives. It satisfies Core Selection and needs less weakening of Cost Monotonicity and Strict Ranking, although this comes at the expense of Population Monotonicity.

Theorem 5 *i) No solution satisfies Cost Monotonicity$_1$ and Core Selection.*
ii) No solution satisfies Strict Ranking$_1$ and Core Selection.

Proof. Consider a three player game where $c_{03} = 1$ and $c_e = 0$ otherwise and c' such that $c'_e = 0$ for all e. Solving, $C(\{3\}, c) = 1$, $C(S, c) = 0$ for all $S \neq \{3\}$ and $C(S, c') = 0$ for all S. By core selection, $y_i(N, c') = 0$ for all $i \in N$. By Core Selection, $y_1(N, c) + y_3(N, c) \leq 0$, $y_2(N, c) + y_3(N, c) \leq 0$ and $y_1(N, c) + y_2(N, c) + y_3(N, c) = 0$. The only point in the core of C is $(0, 0, 0)$.

i) By Cost Monotonicity$_1$, $y_3(N, c) > y_3(N, c') = 0$. Any such allocation violates Core Selection for c.

ii) By Strict Ranking$_1$, $y_1(N, c) < y_3(N, c)$. Any such allocation violates Core Selection for c. ■

We therefore need to choose between Core Selection and the stronger versions of Cost Monotonicity and Strict Ranking.

6 Non-negativity

In most of the literature on cost sharing of mcst problems, Non-negativity is assumed without a closer look at what it implies. We examine the links between Non-negativity and the various properties defined in the previous section.

Theorem 6 *A solution that satisfies Symmetry and Population Monotonicity satisfies Non-negativity.*

Proof. Suppose that $y_i(N, c) < 0$. Take $j \notin N$ and c' such that $c'_e = c_e$ for all $e \in N_0^P$ and $c'_{jk} = c'_{ik}$ for all $k \in N_0$. By Symmetry, $y_i(N \cup \{j\}, c') = y_j(N \cup \{j\}, c')$. By Population Monotonicity, $y_k(N \cup \{j\}, c') \leq y_k(N, c)$ for all $k \in N$. In particular, $y_i(N \cup \{j\}, c') \leq y_i(N, c) < 0$. Therefore, $\sum_{k \in N} y_k(N \cup \{j\}, c') \leq C(N, c)$. Since agents in $N \setminus \{i\}$ do not gain anything from j that they do not gain from i since i and j are symmetric, we have $C(N, c) \leq C(N \cup \{j\}, c')$. It is therefore impossible to have simultaneously $\sum_{k \in N} y_k(N \cup \{j\}, c') \leq C(N, c)$ and $y_j(N \cup \{j\}, c') < 0$. Negative cost shares are incompatible with Symmetry and Population Monotonicity. ■

Therefore, while we argued that Non-negativity is not in itself a desirable property, we can see here that it is implied by Symmetry and Population Monotonicity, two desirable properties.

Theorem 7 *A solution that satisfies Cost Monotonicity$_2$ and Symmetry does not satisfy Non-negativity. A solution that satisfies Strict Ranking$_2$ does not satisfy Non-negativity.*

Proof. Let $N = \{1, 2\}$, c be such that $c_{01} = c_{12} = 0$ and $c_{02} = 1$ and c' be such that $c'_{01} = c'_{02} = c'_{12} = 0$.

By Symmetry, $y_1(N, c') = y_2(N, c') = 0$. By Cost Monotonicity$_2$, $y_2(N, c) > y_2(N, c') = 0$. Since $C(N, c) = C(N, c') = 0$, we must have $y_1(N, c) < y_1(N, c') = 0$.

By Strict Ranking$_2$, $y_1(N, c) < y_2(N, c)$. Since $C(N, c) = 0$, $y_1(N, c) < 0 < y_2(N, c)$. ■

Thus, as soon as we ask for more than the minimum versions of Cost Monotonicity and Strict Ranking, we are limited to solutions that sometimes allocate negative cost shares. Putting the results of this section together, we obtain the following corollary.

Corollary 1 *There are no solutions satisfying Symmetry, Population Monotonicity and Cost Monotonicity$_2$. There are no solutions satisfying Symmetry, Population Monotonicity and Strict Ranking$_2$.*

7 Concluding remarks

As asymmetric solutions are not interesting, we have the result that Population Monotonicity is incompatible with anything more than the weakest versions of Cost Monotonicity and Strict Ranking. Note that this is in contrast with a result obtained in Bogomolnaia et al. (2010) for the related capacity synthesis game, where each pair of agents requests a connection guaranteeing a certain capacity. In their model, they show that Population Monotonicity is compatible with Strict Ranking$_1$ and Symmetry.

Corollary 1, together with Theorem 5, shows that properties related to responsibility towards one's location and properties related to stability are difficult to obtain together, and we must often choose between them. The Kar and folk solutions are extremes, with the former satisfying responsibility properties and the latter satisfying stability properties. Trudeau (2010) shows this choice clearly. However, the new solution y^{CC} offers an interesting compromise; it satisfies Core Selection and stronger responsibility properties than the folk solution.

In the no property rights approach, it is still an open question if we can achieve Core Selection without throwing as much information away as the folk solution, therefore allowing for more than Cost Monotonicity$_3$ and Strict Ranking$_3$.

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